

The cohomology ring of free loop spaces

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Abstract

Let X be a simply connected space and \mathbb{k} a commutative ring. Goodwillie, Burghelea and Fiedorowicz proved that the Hochschild cohomology of the singular chains on the pointed loop space $HH^*S_*(\Omega X)$ is isomorphic to the free loop space cohomology $H^*(X^{S^1})$. We proved that this isomorphism is compatible with both the cup product on $HH^*S_*(\Omega X)$ and on $H^*(X^{S^1})$. In particular, we explicit the algebra $H^*(X^{S^1})$ when X is a suspended space, a complex projective space or a finite CW-complex of dimension p such that $\frac{1}{(p-1)!} \in \mathbb{k}$.

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1 Introduction

Let X be a simply connected CW-complex. We consider the free loop space cohomology algebra of X , $H^*(X^{S^1})$, with coefficients in any arbitrary commutative ring \mathbb{k} .

When \mathbb{k} is a field, the algebra $H^*(X^{S^1})$ can sometimes be computed via spectral sequences. For example, Kuribayashi and Yamaguchi [20], by solving extensions problems by applications of the Steenrod operations, were able to compute via the Eilenberg-Moore spectral sequence, the algebra $H^*(X^{S^1})$ for some simple spaces.

If A is a commutative algebra, the Hochschild homology of A , $HH_*(A)$ can be endowed with the shuffle product. When $\mathbb{k} = \mathbb{Q}$, Sullivan, inventing Rational Homotopy, constructed a commutative algebra $A_{PL}(X)$, called the *polynomial differential forms* [14, §10], and proved with Vigué-Poirrier [32] that there is a natural isomorphism of graded algebras

$$H^*(X^{S^1}) \cong HH_*(A_{PL}(X)).$$

The theory of minimal Sullivan models gives a technic to compute the algebra $HH_*(A_{PL}(X))$ and so $H^*(X^{S^1})$.

Denote by $HH^*(A)$ the Hochschild cohomology of an algebra A with coefficients in $A^\vee = \text{Hom}(A, \mathbb{k})$ [23, 1.5.5]. If A has a diagonal $\Delta : A \rightarrow A \otimes A$, then $HH^*(A)$ is equipped with a cup product. The normalized singular chains on the pointed loop space $S_*(\Omega X)$ is a differential graded Hopf algebra. So the Hochschild cohomology $HH^*(S_*(\Omega X))$ is naturally a graded algebra. In 1985, Goodwillie [15], Burghelea and Fiedorowicz [7] proved that there is an isomorphism of graded modules

$$HH^*(S_*(\Omega X)) \cong H^*(X^{S^1}).$$

The main result of this paper is that this isomorphism of graded modules is, in fact, an isomorphism of graded algebras (Theorem 3.1). This result gives a general method to compute the algebra $H^*(X^{S^1})$ over any commutative ring:

- Determine the Adams-Hilton model [1] of X , $\mathcal{A}(X)$, using its cellular decomposition.
- Compute the structure of Hopf algebra up to homotopy model on $\mathcal{A}(X)$ (Definition 4.3), knowing the structure of graded Hopf algebra of $H_*(\Omega X)$ when \mathbb{k} is a field.

Note that, if X is an H-space, our approach has no interest, since simply

$$X^{S^1} \approx \Omega X \times X.$$

Now, Theorem 4.6 allows us to replace, in the Hochschild cohomology, the differential graded Hopf algebra $S_*(\Omega X)$ by the Adams-Hilton model of X , $\mathcal{A}(X)$, equipped with its structure of Hopf algebras up to homotopy.

We face now (section 5) a completely algebraic problem: How to compute the Hochschild cohomology on a Hopf algebra up to homotopy whose underlying algebra is a tensor algebra.

In section 6, we show that in a simple case, this Hochschild cohomology reduces to the Hochschild homology of a commutative algebra.

In section 7, we investigated the algebra structure of the free loop space cohomology on any suspension ΣX . If $H_*(X)$ is \mathbb{k} -free of finite type, this algebra $H^*((\Sigma X)^{S^1})$ is the Hochschild homology of $H^*(\Sigma X)$, $HH_*(H^*(\Sigma X))$, equipped with a product completely determined by the cohomology algebra of the desuspended space, $H^*(X)$. Recall from [22] and [29] that, even when $\mathbb{k} = \mathbb{Q}$ and ΣX is a wedge of spheres, the cohomology algebra $H^*((\Sigma X)^{S^1})$ is particularly difficult to explicit in terms of generators and relations.

In section 8, we prove that the free loop space cohomology on the complex projective space \mathbb{CP}^n , $H^*((\mathbb{CP}^n)^{S^1})$, is isomorphic as graded algebras to the Hochschild homology $HH_*(H^*(\mathbb{CP}^n))$ and compute it. Suppose that X is a finite CW-complex of dimension p such that $\frac{1}{(p-1)!} \in \mathbb{k}$. Anick [2, dualize Proposition 8.7(a)], extending Sullivan's result, constructed a commutative algebra $\mathcal{C}^*(\mathbf{L}(X))$ weakly equivalent as algebras (in the sense of [13, page 832]) to the singular cochains on X , $S^*(X)$. We extend Sullivan and Vigué-Poirrier result in this new context (Theorem 8.4).

We would like to mention that Ndbol and Thomas [4] have also found, when \mathbb{k} is a field, a general method to compute the free loop space cohomology algebra of a simply connected space. We thank S. Halperin, J.-C. Thomas and M. Vigué for their constant support. The main results of this paper were exposed in September 1999, at the GDR Topologie algébrique meeting in Paris Nord.

2 Algebraic preliminaries and notation

We work over a commutative ring \mathbb{k} . We denote by ${}_p\mathbb{k}$ and $\frac{\mathbb{k}}{p\mathbb{k}}$ respectively the kernel and cokernel of the multiplication by p in \mathbb{k} .

DGA stands for differential graded algebra, DGC for differential graded coalgebra, DGH for differential graded Hopf algebra and CDGA for commutative DGA. The denomination “chain” will be restricted to objects with a non-negative lower degree and “cochain” to those with a non-negative upper degree.

The degree of an element x is denoted $|x|$. The *suspension* of a graded module V is the graded module sV such that $(sV)_{i+1} = V_i$. Let C be an augmented complex. The kernel of the augmentation is denoted \overline{C} .

The exterior algebra on an element v is denoted Ev . The *free divided powers algebra* on an element v , denoted Γv , is

- the free graded algebra generated by $\gamma^i(v)$, $i \in \mathbb{N}^*$, divided by the relations $\gamma^i(v)\gamma^j(v) = \frac{(i+j)!}{i!j!}\gamma^{i+j}(v)$, if $|v|$ is even,
- and is just Ev when $|v|$ is odd.

The *tensor algebra* on a graded module V is denoted TAV . The *tensor coalgebra* is denoted TCV . Their common underlying module is simply denoted TV . Given a conilpotent coalgebra C then any morphism $\varphi : \overline{C} \rightarrow V$ lifts uniquely to a unique morphism $\Psi : C \rightarrow TCV$ of coaugmented coalgebras. The formula for Ψ is given by

$$\Psi(c) = \sum_{i=1}^{+\infty} \varphi^{\otimes i} \circ \overline{\Delta}_C^{\otimes i-1}(c), c \in \overline{C} \quad (2.1)$$

where $\Delta_C^{i-1} : \overline{C} \rightarrow \overline{C}^{\otimes i}$ is the iterated reduced diagonal of C .

Let A be an augmented DGA. Denote d_1 be the differential of the complex $A \otimes T(s\overline{A}) \otimes A$ obtained by tensorization. We denote the tensor product of the elements $a \in A$, $sa_1 \in s\overline{A}$, \dots , $sa_k \in s\overline{A}$ and $b \in A$ by $a[sa_1|\dots|sa_k]b$. Let d_2 be the differential on the graded module $A \otimes T(s\overline{A}) \otimes A$ defined by:

$$\begin{aligned} d_2 a[sa_1|\dots|sa_k]b &= (-1)^{|a|} aa_1[sa_2|\dots|sa_k]b \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} a[sa_1|\dots|sa_i a_{i+1}|\dots|sa_k]b \\ &\quad - (-1)^{\varepsilon_{k-1}} a[sa_1|\dots|sa_{k-1}]a_k b; \end{aligned}$$

Here $\varepsilon_i = |a| + |sa_1| + \cdots + |sa_i|$.

The *bar resolution of A* , denoted $B(A; A; A)$, is the (A, A) -bimodule $(A \otimes T(s\bar{A}) \otimes A, d_1 + d_2)$. The *(reduced) bar construction on A* , denoted $B(A)$, is the coaugmented DGC $(TCs\bar{A}, d_1 + d_2)$ whose underlying complex $(Ts\bar{A}, d_1 + d_2)$ coincides with $\mathbb{k} \otimes_A B(A; A; A) \otimes_A \mathbb{k}$ [13, §4]. The *cyclic bar construction* or *Hochschild complex* is the complex $A \otimes_{A \otimes A^{op}} B(A; A; A)$ denoted $C(A)$. Explicitly $C(A)$ is the complex $(A \otimes T(s\bar{A}), d_1 + d_2)$ with d_1 obtained by tensorization and

$$\begin{aligned} d_2 a[sa_1 | \cdots | sa_k] &= (-1)^{|a|} a a_1[sa_2 | \cdots | sa_k] \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} a[sa_1 | \cdots | sa_i a_{i+1} | \cdots | sa_k] \\ &\quad - (-1)^{|sa_k| \varepsilon_{k-1}} a_k a[sa_1 | \cdots | sa_{k-1}]; \end{aligned}$$

The *Hochschild homology* is the homology of the cyclic bar construction:

$$HH_*(A) := H_*(C(A)).$$

The *Hochschild cohomology* is the graded module

$$HH^*(A) := H^*(\text{Hom}_{(A,A)}(B(A; A; A), A^\vee)) = H^*(C(A)^\vee)$$

where A^\vee is considered as an (A, A) -bimodule.

Let A and B be two augmented DGA's, Then we have an Alexander-Whitney morphism of $(A \otimes B, A \otimes B)$ -bimodules

$$AW : B(A \otimes B; A \otimes B; A \otimes B) \rightarrow B(A; A; A) \otimes B(B; B; B)$$

where the image of a typical element $p \otimes q[s(a_1 \otimes b_1) | \cdots | s(a_k \otimes b_k)]m \otimes n$ is

$$\sum_{i=0}^k (-1)^{\zeta_i} p[sa_1 | \cdots | sa_i] a_{i+1} \cdots a_k m \otimes q b_1 \cdots b_i [sb_{i+1} | \cdots | sb_k] n.$$

Here [26, 3.7]

$$\begin{aligned} \zeta_i &= \sum_{j=1}^k \left(|q| + \sum_{l=1}^{j-1} |b_l| \right) |a_j| + \left(|q| + \sum_{j=1}^k |b_j| \right) |m| \\ &\quad + \sum_{j=i+1}^k (j-i) |a_j| + (k-i) |m| + |i| |q| + \sum_{j=1}^{i-1} (i-j) |b_j|. \end{aligned}$$

AW is natural and associative exactly. It is also commutative up to a homotopy of $(A \otimes B, A \otimes B)$ -bimodules. So we get an Alexander-Whitney map for the cyclic bar construction

$$AW : C(A \otimes B) \rightarrow C(A) \otimes C(B).$$

Consider an augmented DGA K equipped with a morphism of augmented DGA's $\Delta : K \rightarrow K \otimes K$. Then the composite

$$\Delta : C(K) \xrightarrow{C(\Delta)} C(K \otimes K) \xrightarrow{AW} C(K) \otimes C(K)$$

is a morphism of augmented complex. Therefore $HH^*(K)$ has a product. This is the cup product of Cartan and Eilenberg [8, XI.6]. In particular, if K is a DGH then $C(K)$ is a DGC and $HH^*(K)$ is a graded algebra.

3 From the chains on the based loops to the chains on the free loops

The object of this section is to prove the following theorem linking the chains on the based loops of a space to the chains on its free loops.

Theorem 3.1 (*Compare [15] and [7]*) *Let X be a path connected pointed space. Then there is a natural DGC quasi-isomorphism*

$$C(S_*(\Omega X)) \xrightarrow{\cong} S_*(X^{S^1}).$$

*In particular, $HH^*S_*(\Omega X) \cong H^*(X^{S^1})$ as graded algebras.*

Goodwillie [15], Burghelea and Fiedorowicz [7] proved the isomorphism $HH^*S_*(\Omega X) \cong H^*(X^{S^1})$ as graded modules only. To obtain our theorem, we will follow their proofs. We introduce first some terminology about simplicial objects.

Let \mathcal{C} be a category. A *simplicial \mathcal{C} -object* X is a non-negative graded object together with morphisms $d_i : X_n \rightarrow X_{n-1}$ and $s_i : X_n \rightarrow X_{n+1}$, $0 \leq i \leq n$ satisfying some well-known relations [24, VIII.5.2]. A *cosimplicial \mathcal{C} -object* is a non-negative graded object together with morphisms $\delta_i : X^{n-1} \rightarrow X^n$ and $\sigma_i : X^{n+1} \rightarrow X^n$, $0 \leq i \leq n$ satisfying the opposite relations [6, X.2.1(i)]. If \mathcal{C} is a category equipped with a tensor product \otimes (more precisely

a monoidal category [25, VII.1]) then the tensor product of two simplicial \mathcal{C} -objects $X = (X_n, d_i, s_i)$ and $Y = (Y_n, d_i, s_i)$ is the simplicial \mathcal{C} -object $X \otimes Y = (X_n \otimes Y_n, d_i \otimes d_i, s_i \otimes s_i)$.

Consider \mathcal{C} to be the category of complexes. To any simplicial \mathcal{C} -object (i.e. simplicial complex) X , we can associate a complex in the category \mathcal{C} (i.e. a complex of complexes) denote $K_N(X)$ known as the normalized chain complex of X [24, VIII.6 for the category of modules]. Consider two simplicial complexes A and B . We have an Alexander-Whitney morphism of complexes of complexes [24, VIII.8.6] $AW : K_N(A \otimes B) \rightarrow K_N(A) \otimes K_N(B)$. Every complex of complexes can be condensated [24, X.9.1] into a single complex. So by composing the functor K_N and the condensation functor, we have a functor, called the *realization* and denoted $|\cdot|$, from the category of simplicial complexes to the category of complexes, equipped with an Alexander-Whitney morphism of complexes $AW : |A \otimes B| \rightarrow |A| \otimes |B|$ for any simplicial complexes A and B . In particular, $|\cdot|$ induces a functor from the category of simplicial DGC's to the category of DGC's (Recall that a simplicial DGC can be defined either as a simplicial object in the category of DGC's or as a coalgebra in the category of simplicial complexes.).

Given any two topological spaces X and Y , the caligraphic notations

$$AW : S_*(X \times Y) \rightarrow S_*(X) \otimes S_*(Y) \text{ and } \mathcal{EZ} : S_*(X) \otimes S_*(Y) \rightarrow S_*(X \times Y)$$

are reserved to the standart normalized Alexander-Whitney map and to the standart normalized Eilenberg-Zilber map concerning singular chains [10, VI.12.27-8].

Example 3.2 The cyclic bar construction for differential graded algebras. Let A be a DGA. Then there is a simplicial complex ΓA defined by $\Gamma_n A = A \otimes \cdots \otimes A = A^{\otimes n+1}$,

$$\begin{aligned} d_0 a[a_1 | \cdots | a_n] &= a a_1 [a_2 | \cdots | a_n], \\ d_i a[a_1 | \cdots | a_n] &= a[a_1 | \cdots | a_i a_{i+1} | \cdots | a_n] \text{ for } 1 \leq i \leq n-1, \\ d_n a[a_1 | \cdots | a_n] &= a_n a[a_1 | \cdots | a_{n-1}], \\ s_i a[a_1 | \cdots | a_n] &= a[a_1 | \cdots | a_i | 1 | a_{i+1} | \cdots | a_n] \text{ for } 0 \leq i \leq n. \end{aligned}$$

The complex $|\Gamma A|$ is exactly (signs included) $C(A)$ the cyclic bar construction of A . If K is a DGH then ΓK with the diagonal

$$\Gamma K \xrightarrow{\Gamma(\Delta_K)} \Gamma(K \otimes K) \cong \Gamma K \otimes \Gamma K$$

is a simplicial DGC and $|\Gamma K|$ is the DGC $C(K)$ denoted in section 2.

Example 3.3 Let G be a topological monoid. The cyclic bar construction of G [23, 7.3.10] is the simplicial space ΓG defined by $\Gamma_n G = G \times \cdots \times G = G^{n+1}$ and with the same formulas for d_i and s_i as in the cyclic bar construction for DGA's. Since the normalized singular chain functor S_* is a functor from topological spaces to DGC's, $S_*(\Gamma G)$ is a simplicial DGC. Therefore $|S_*(\Gamma G)|$ is a DGC.

The following Lemma compares the DGC's given by the previous two examples.

Lemma 3.4 (*Compare [15, V.1.2]*) *Let G be a topological monoid. Then there is a natural DGC quasi-isomorphism $|\Gamma S_*(G)| \xrightarrow{\cong} |S_*(\Gamma G)|$.*

Proof. The Eilenberg-Zilber map $\mathcal{EZ} : S_*(G)^{\otimes n+1} \rightarrow S_*(G^{n+1})$ is a DGC quasi-isomorphism and therefore defines a morphism of simplicial DGC's $\Gamma S_*(G) \rightarrow S_*(\Gamma G)$. So, applying the functor $|\cdot|$, we get a DGC quasi-isomorphism. QED

Let Δ^n be the standard geometric simplex of dimension n . Let $\delta_i : \Delta^{n-1} \rightarrow \Delta^n$ and $\sigma_i : \Delta^{n+1} \rightarrow \Delta^n$ be the i -th face inclusion and the i -th degeneracy of Δ^n . Then $\Delta = (\Delta^n, \delta_i, \sigma_i)$ is a cosimplicial space [6, X.2.2(i)]. The geometric realization [27, 11.1] of a simplicial space X is defined as

$$|X| = \left(\coprod_{n \in \mathbb{N}} X_n \times \Delta^n \right) / \sim$$

where \sim is the equivalence relation generated by

$$(d_i x, y) \sim (x, \delta_i y), \quad x \in X_n, \quad y \in \Delta^{n-1}$$

$$\text{and } (s_i x, y) \sim (x, \sigma_i y), \quad x \in X_n, \quad y \in \Delta^{n+1}.$$

Recall that $|S_*(X)|$ is a DGC whose diagonal is the composite

$$|S_*(X)| \xrightarrow{|S_*(\Delta)|} |S_*(X \times X)| \xrightarrow{|\mathcal{AW}|} |S_*(X) \otimes S_*(X)| \xrightarrow{AW} |S_*(X)| \otimes |S_*(X)|$$

Lemma 3.5 (Compare [5, Theorem 4.1] and [14, 17(a)]) *Let X be a simplicial space, good in the sense of [31, A.4]. Then there is a natural DGC quasi-isomorphism $f : |S_*(X)| \xrightarrow{\sim} S_*(|X|)$.*

Proof. Let $\pi_n : X_n \times \Delta^n \twoheadrightarrow |X|$ be the quotient map. The morphism f is defined as the composite

$$S_i(X_n) \xrightarrow{id_{S_i(X_n)} \otimes \kappa_n} S_i(X_n) \otimes S_n(\Delta^n) \xrightarrow{\mathcal{EZ}} S_{i+n}(X_n \times \Delta^n) \xrightarrow{S_{i+n}(\pi_n)} S_{n+i}(|X|)$$

where $\kappa_n \in S_n(\Delta^n)$ is the singular simplex id_{Δ^n} . We just have to prove that f is a DGC morphism. The diagonal map of $|X|$ is equal to the composite

$$|X| \xrightarrow{|\Delta_X|} |X \times X| \xrightarrow{(|proj_1|, |proj_2|)} |X| \times |X|$$

where $|\Delta_X|$ is the simplicial diagonal of X and where $proj_1$ and $proj_2$ are the simplicial projections on each factors. So by naturality of f , it suffices to show that f well behaves with products of simplicial spaces.

Let X and Y be two simplicial spaces. Then $S_*(X)$ and $S_*(Y)$ are two simplicial complexes. So we have an Alexander-Whitney map

$$|S_*(X) \otimes S_*(Y)| \xrightarrow{AW} |S_*(X)| \otimes |S_*(Y)|.$$

Its formula is given by

$$\left[\sum_{p+q=n} S_*(\tilde{d}^q) \otimes S_*(d_0^p) \right] : S_*(X_n) \otimes S_*(Y_n) \longrightarrow \bigoplus_{p+q=n} S_*(X_p) \otimes S_*(Y_q)$$

where $\tilde{d}^q : X_n \rightarrow X_p$ is the composite $d_{p+1} \circ \dots \circ d_n$ (\tilde{d} denotes the “last” face operator) and $d_0^p : Y_n \rightarrow Y_q$ is the iterated composite of d_0 . In the diagram page 10, there was no space left for sums \sum and direct sums \bigoplus . So we use the indices p and q with the convention $p + q = n$ and the indices j and k with the conventions that $j + k = i$. We use also the maps

$$\begin{aligned} \tilde{\delta}^q &= \delta_n \circ \dots \circ \delta_{p+1} : \Delta^p \rightarrow \Delta^n, & \delta_0^p &: \Delta^q \rightarrow \Delta^n, \\ \tilde{\sigma}_0^q &= \sigma_p \circ \dots \circ \sigma_{n-1} : \Delta^n \rightarrow \Delta^p & \text{and } \sigma_0^p &: \Delta^n \rightarrow \Delta^q. \end{aligned}$$

Both the interchange of factors of a tensor product of modules and of a product of spaces are denoted by τ .

Consider the diagram page 10.

$$\begin{array}{ccccccc}
S_i(X_n \times Y_n) & \xrightarrow{id \otimes \kappa_n} & S_i(X_n \times Y_n) \otimes S_n(\Delta^n) & \xrightarrow{\mathcal{E}\mathcal{Z}} & S_{n+i}(X_n \times Y_n \times \Delta^n) & \xrightarrow{S_{n+i}(\pi_n)} & S_{n+i}(|X \times Y|) \\
\downarrow \mathcal{AW} & & \downarrow id \otimes S_n(\Delta) & \textcircled{3} & \downarrow S_{n+i}(id \times \Delta) & & \downarrow S_{n+i}(|proj_1|, |proj_2|) \\
& & S_i(X_n \times Y_n) \otimes S_n(\Delta^n \times \Delta^n) & \xrightarrow{\mathcal{E}\mathcal{Z}} & S_{n+i}(X_n \times Y_n \times \Delta^n \times \Delta^n) & \textcircled{6} & \\
& & \downarrow \mathcal{AW} \otimes \mathcal{AW} & \textcircled{4} & \downarrow S_{n+i}(id \times \tau \times id) & & \\
S_j(X_n) \otimes S_k(Y_n) & \xrightarrow{id \otimes \mathcal{AW}(\kappa_n, \kappa_n)} & S_j(X_n) \otimes S_k(Y_n) \otimes S_p(\Delta^n) \otimes S_q(\Delta^n) & & S_{n+i}(X_n \times \Delta^n \times Y_n \times \Delta^n) & \xrightarrow{S_{n+i}(\pi_n \times \pi_n)} & S_{n+i}(|X| \times |Y|) \\
\downarrow S_j(\tilde{d}^q) \otimes S_k(d_0^p) & \textcircled{2} & \downarrow S_j(\tilde{d}^q) \otimes S_k(d_0^p) \otimes S_p(\tilde{\sigma}^q) \otimes S_q(\sigma_0^p) & \searrow \mathcal{E}\mathcal{Z} \otimes \mathcal{E}\mathcal{Z} \circ (id \otimes \tau \otimes id) & \downarrow \mathcal{AW} & \textcircled{7} & \downarrow \mathcal{AW} \\
S_j(X_p) \otimes S_k(Y_q) & \xrightarrow{id \otimes \kappa_p \otimes \kappa_q} & S_j(X_p) \otimes S_k(Y_q) \otimes S_p(\Delta^p) \otimes S_q(\Delta^q) & \textcircled{5} & S_{p+j}(X_n \times \Delta^n) \otimes S_{q+k}(Y_n \times \Delta^n) & \xrightarrow{S_{p+j}(\pi_n) \otimes S_{q+k}(\pi_n)} & S_{p+j}(|X|) \otimes S_{q+k}(|Y|) \\
& & \downarrow & \searrow \mathcal{E}\mathcal{Z} \otimes \mathcal{E}\mathcal{Z} \circ (id \otimes \tau \otimes id) & \downarrow S_{p+j}(\tilde{d}^q \times \tilde{\sigma}^q) \otimes S_{q+k}(d_0^p \times \sigma_0^p) & \textcircled{8} & \downarrow S_{p+j}(\pi_p) \otimes S_{q+k}(\pi_q) \\
& & S_j(X_p) \otimes S_k(Y_q) \otimes S_p(\Delta^p) \otimes S_q(\Delta^q) & & S_{p+j}(X_p \times \Delta^p) \otimes S_{q+k}(Y_q \times \Delta^q) & &
\end{array}$$

Let's check the commutativity of each subdiagram involved in it.

- 1 commutes obviously since $S_n(\Delta)(\kappa_n) = (\kappa_n, \kappa_n)$.
- 2 commutes since

$$\begin{aligned} \left[\bigoplus_{p+q=n} S_p(\tilde{\sigma}^q) \otimes S_q(\sigma_0^p) \right] \circ \mathcal{AW}(\kappa_n, \kappa_n) &= \sum_{p+q=n} S_p(\tilde{\sigma}^q) \otimes S_q(\sigma_0^p)(\tilde{\delta}^q \otimes \delta_0^p) \\ &= \sum_{p+q=n} \kappa_p \otimes \kappa_q. \end{aligned}$$

- 3 commutes by naturality of \mathcal{EZ} .
- 4 commutes by compatibility of \mathcal{EZ} and \mathcal{AW} [14, I.4.b)].
- 5 commutes by naturality of $(\mathcal{EZ} \otimes \mathcal{EZ}) \circ (id \otimes \tau \otimes id)$.
- 6 commutes since

$$(|proj_1|, |proj_2|) \circ \pi_n = (\pi_n \times \pi_n) \circ (id \times \tau \times id) \circ (id \times \Delta).$$

- 7 commutes by naturality of \mathcal{AW} .
- 8 does not commute. But the two different maps coming from 8 coincide on the image of $(\mathcal{EZ} \otimes \mathcal{EZ}) \circ (id \otimes \tau \otimes id) \circ [id \otimes \mathcal{AW}(\kappa_n, \kappa_n)]$. Indeed this image is embedded in the image of

$$\sum_{p+q=n} S_*(id_{X_n} \times \tilde{\delta}^q) \otimes S_*(id_{Y_n} \times \delta_0^p).$$

Now

$$\pi_p \circ (\tilde{d}^q \times \tilde{\sigma}^q) \circ (id_{X_n} \times \tilde{\delta}^q) = \pi_p \circ (\tilde{d}^q \times id_{\Delta^p}) = \pi_n \circ (id_{X_n} \times \tilde{\delta}^q).$$

We have a similar formula for Y_n .

Finally, we have

$$(f \otimes f) \circ \left[\sum_{p,q} S_*(\tilde{d}^q) \otimes S_*(d_0^p) \right] \circ |\mathcal{AW}| = \mathcal{AW} \circ S_*((|proj_1|, |proj_2|) \circ f).$$

QED

Lemma 3.6 [23, 7.3.15] *Let X be a path connected pointed space. Then there is a natural homotopy equivalence $|\Gamma\Omega X| \xrightarrow{\cong} X^{S^1}$.*

Proof of Theorem 3.1 Applying Lemma 3.4 to the Moore loop space ΩX , Lemma 3.5 to $\Gamma\Omega X$ and Lemma 3.6 yield to the sequence of DGC quasi-isomorphisms:

$$CS_*(\Omega X) = |\Gamma S_*(\Omega X)| \xrightarrow{\cong} |S_*(\Gamma\Omega X)| \xrightarrow{\cong} S_*(|\Gamma\Omega X|) \xrightarrow{\cong} S_*(X^{S^1}).$$

QED

4 HAH models

In order to compute the algebra structure of $HH^*S_*(\Omega X)$, it is necessary to replace $S_*(\Omega X)$ by a smaller Hopf algebra. Let's first remark that the cyclic bar construction preserves quasi-isomorphisms.

Property 4.1 [23, 5.3.5](Compare [13, 4.3(iii)]) *Let $f : A \rightarrow B$ be a quasi-isomorphism of augmented DGA's. If \overline{A} and \overline{B} are \mathbb{k} -semifree then $C(f) : C(A) \xrightarrow{\cong} C(B)$ is a quasi-isomorphism of complexes.*

Let $f, g : A \rightarrow B$ be two morphisms of augmented DGA's. A *derivation homotopy* from f to g is a morphism of graded modules of degree $+1$, $h : A \rightarrow \overline{B}$ such that $d \circ h + h \circ d = f - g$ and $h(xy) = h(x)g(y) + (-1)^{|x|}f(x)h(y)$ for $x, y \in A$. A derivation homotopy from f to g is denoted by $h : f \approx g$. We say that f and g are *homotopic* if there is a derivation homotopy between them.

Lemma 4.2 *Let $f, g : A \rightarrow B$ be two morphisms of augmented DGA's. If f and g are homotopic then the morphisms of complexes $C(f), C(g) : C(A) \rightarrow C(B)$ are chain homotopic.*

Proof. Let h be a derivation homotopy from f to g . By induction on the wordlength of the cyclic bar construction, construct an explicit chain homotopy between $C(f)$ and $C(g)$. QED

A *Hopf algebra up to homotopy*, or HAH, is a DGA K equipped with two morphisms of DGA's $\Delta : K \rightarrow K \otimes K$ and $\varepsilon : K \rightarrow \mathbb{k}$ such that $(\varepsilon \otimes id_K) \circ \Delta = id_K = (id_K \otimes \varepsilon) \circ \Delta$ (counitary exactly), $(\Delta \otimes 1) \circ \Delta \approx (1 \otimes \Delta) \circ \Delta$ (coassociative up to homotopy) and $\tau \circ \Delta \approx \Delta$ (cocommutative up to homotopy).

Let K, K' be two HAH's. A morphism of augmented DGA's $f : K \rightarrow K'$ is a *HAH morphism* if $\Delta f \approx (f \otimes f)\Delta$ (f commutes with the diagonals up to homotopy).

Definition 4.3 Let X be a pointed topological space. A *HAH model* for X is a free chain algebra (TAV, ∂) equipped with a structure of Hopf algebras up to homotopy and with a HAH quasi-isomorphism $\Theta : (\text{TAV}, \partial) \xrightarrow{\simeq} S_*(\Omega X)$.

The existence of HAH models for any space is guarantied by the following two properties.

Property 4.4 [13, 3.1] Let A be a chain algebra. There exists a free chain algebra (TAV, ∂) and a quasi-isomorphism of augmented chain algebras

$$\Theta : (\text{TAV}, \partial) \xrightarrow{\simeq} A.$$

Property 4.5 [3, I.7 and II.1.11=II.2.11a)] or [13, 3.6] Consider a quasi-isomorphism of augmented chain algebras $p : A \xrightarrow{\simeq} B$ and a morphism of augmented chain algebras g from a free chain algebra (TAV, ∂) to B :

$$\begin{array}{ccc} & & A \\ & \nearrow f & \downarrow p \\ (\text{TAV}, \partial) & \xrightarrow{g} & B \end{array}$$

Then

- i) there is a morphism of augmented chain algebras $f : (\text{TAV}, \partial) \rightarrow A$ such that $p \circ f \approx g$,
- ii) moreover, any two such morphisms f are homotopic.

Indeed by Property 4.4, we obtain a quasi-isomorphism of augmented DGA's

$$\Theta : (\text{TAV}, \partial) \xrightarrow{\simeq} S_*(\Omega X).$$

Since (TAV, ∂) and $S_*(\Omega X)$ are \mathbb{k} -semifree ¹, $\Theta \otimes \Theta$ is a quasi-isomorphism. By Property 4.5 i), we obtain a diagonal Δ_{TAV} for (TAV, ∂) such that the

¹We do not assume that \mathbb{k} is a principal ideal domain [13, §2].

following diagram of augmented DGA's commutes up to homotopy:

$$\begin{array}{ccc}
(\mathrm{TAV}, \partial) & \xrightarrow[\Theta]{\simeq} & S_*(\Omega X) \\
\Delta_{\mathrm{TAV}} \downarrow & & \downarrow \Delta_{S_*(\Omega X)} \\
(\mathrm{TAV}, \partial) \otimes (\mathrm{TAV}, \partial) & \xrightarrow[\Theta \otimes \Theta]{\simeq} & S_*(\Omega X) \otimes S_*(\Omega X)
\end{array}$$

Since $S_*(\Omega X)$ is exactly a DGH and is cocommutative up to homotopy, by Property 4.5 ii), Δ is counitary, coassociative and cocommutative up to homotopy. The diagonal Δ can be chosen to be strictly counitary [2, Lemma 5.4].

Theorem 4.6 *Let X be a path connected pointed space. Let (TAV, ∂) be a HAH model for X . There is a isomorphism of graded algebras*

$$HH^*(\mathrm{TAV}, \partial) \cong H^*(X^{S^1}).$$

Proof. Let $\Theta : (\mathrm{TAV}, \partial) \xrightarrow{\sim} S_*(\Omega X)$ denote the HAH quasi-isomorphism. By Property 4.1, $C(\Theta)$ is a quasi-isomorphism. According Lemma 4.2,

$$C((\Theta \otimes \Theta) \circ \Delta_{\mathrm{TAV}}) \approx C(\Delta_{S_*(\Omega X)} \circ \Theta).$$

So by composing with AW and by applying Theorem 3.1, the quasi-isomorphisms of chain complexes

$$C(\mathrm{TAV}, \partial) \xrightarrow{C(\Theta)} CS_*(\Omega X) \xrightarrow{\sim} S_*(X^{S^1})$$

commutes with the diagonals up to chain homotopy. \square

5 A smaller resolution than the bar resolution

The goal of this section is to replace the huge algebra up to homotopy $C(\mathrm{TAV}, \partial)^\vee$ by a smaller in order to be able to compute the algebra $HH^*(\mathrm{TAV}, \partial)$. When \mathbb{k} is a field, Micheline Vigué in [33] gives a small complex $((\mathbb{k} \oplus sV) \otimes \mathrm{TV}, \delta)$ whose homology is the vector space $HH_*(\mathrm{TAV}, \partial)$. In fact, in this section, we show that over any commutative ring \mathbb{k} , this complex $((\mathbb{k} \oplus sV) \otimes \mathrm{TV}, \delta)$ is a strong deformation retract of $C(\mathrm{TAV}, \partial)$.

Definition 5.1 Let (Y, d) be a complex. A complex (X, ∂) is a *strong deformation retract* of (Y, d) if there exist two morphisms of complexes $\nabla : (X, \partial) \hookrightarrow (Y, d)$, $f : (Y, d) \rightarrow (X, \partial)$ and a chain homotopy $\Phi : (Y, d) \rightarrow (Y, d)$ such that $f\nabla = id_X$ and $\nabla f - id_Y = d\Phi + \Phi d$. The map f is called the *projection* and the map ∇ is called the *inclusion*.

We first consider the case where the differential ∂ on (TAV, ∂) is just obtained by tensorization of the differential of a complex V and is so therefore homogeneous by wordlength.

Consider the tensor algebra TAV on a complex V . Define the augmentation on TAV such that the augmentation ideal \overline{TAV} is

$$T^+V = \oplus_{i \geq 1} V^{\otimes i}.$$

The bar resolution $B(TAV; TAV; TAV)$ contains a subcomplex $(TV \otimes (\mathbb{k} \oplus sV) \otimes TV, d_1 + d_2)$, since

$$d_2(a \otimes sv \otimes b) = (-1)^{|a|}(av \otimes b - a \otimes vb).$$

Proposition 5.2 [23, Proposition 3.1.2] *The (TAV, TAV) -bimodule $(TV \otimes (\mathbb{k} \oplus sV) \otimes TV, d_1 + d_2)$ is a strong deformation retract of the bar resolution $B(TAV; TAV; TAV)$.*

Proof. Define the projection $f : B(TAV; TAV; TAV) \rightarrow TV \otimes (\mathbb{k} \oplus sV) \otimes TV$ on its components $f_n : TV \otimes (sT^+V)^{\otimes n} \otimes TV \rightarrow TV \otimes (\mathbb{k} \oplus sV) \otimes TV$:

The map $f_0 : TV \otimes TV \rightarrow TV \otimes TV$ is the identity map.

We define $f_1 : TV \otimes sT^+V \otimes TV \rightarrow TV \otimes sV \otimes TV$ by

$$f_1(a[sv_1 \cdots v_n]b) = \sum_{i=1}^n (-1)^{|v_1 \cdots v_{i-1}|} av_1 \cdots v_{i-1} \otimes sv_i \otimes v_{i+1} \cdots v_nb$$

for $a, b \in TV$, $v_1, \dots, v_n \in V$ and $n \in \mathbb{N}^*$.

For $n \geq 2$, f_n is the zero map. An easy calculation shows that $d_2 f_1 = f_0 d_2$. Since $f_1(a[sa_1 a_2]b) = f_1(a[sa_1]a_2 b) + (-1)^{|a_1|} f_1(aa_1[sa_2]b)$, $f_1 d_2 = 0$. Therefore f commutes with d_2 and is a morphism of complexes.

Of course, $f\nabla = id_{TV \otimes (\mathbb{k} \oplus sV) \otimes TV}$. The components

$$\Phi_n : TV \otimes (sT^+V)^{\otimes n} \otimes TV \rightarrow TV \otimes (sT^+V)^{\otimes n+1} \otimes TV$$

of the chain homotopy Φ are defined by:

$$\begin{aligned}\Phi_0 &= 0, \\ \Phi_n(a[sa_1|\cdots|sa_{n-1}|sv]b) &= 0, \\ \Phi_n(a[sa_1|\cdots|sa_{n-1}|sa_nv]b) &= -(-1)^{\varepsilon_n}a[sa_1|\cdots|sa_n|sv]b \\ &\quad + \Phi_n(a[sa_1|\cdots|sa_n]vb)\end{aligned}$$

for $a, b \in TV$, $v \in V$ and $a_1, \dots, a_n \in T^+V$. Recall that $\varepsilon_n = |a| + |sa_1| + \dots + |sa_n|$.

By a double induction first on n and then on the wordlength, check that $d\Phi_n + \Phi_{n-1}d = \nabla f_n - id$, $n \in \mathbb{N}$. At the beginning for $n = 1$, use the formula $f_1(a[sa_1v]b) = f_1(a[sa_1]vb) + (-1)^{|a_1|}aa_1 \otimes sv \otimes b$. \boxed{QED}

Consider now an augmented DGA (TAV, ∂) such that $\overline{TAV} = T^+V$. The differential ∂ decomposes uniquely as a sum $d_1 + d_2 + \dots + d_i + \dots$ of derivations satisfying $d_i(V) \subset T^iV = V^{\otimes i}$. The differential d_1 is called the *linear part* of d .

To pass from the case $\partial = d_1$ to the general case, we'll use the well-known perturbation Lemma. For an abundant and recent bibliography, see [21] or [18].

Theorem 5.3 (*Perturbation Lemma*) *Let $(X, \partial) \xrightleftharpoons[\nabla]{f} (Y, d) \circlearrowleft \Phi$ be a strong deformation retract of chain complexes satisfying $f\Phi = 0$, $\Phi\nabla = 0$ and $\Phi^2 = 0$. Suppose moreover that this strong deformation retract is filtered: there exist on X and on Y increasing filtrations bounded below preserved by ∂ , d , f , ∇ and Φ . Consider a filtration lowering linear map $t : Y \rightarrow Y$ of degree -1 such that $d + t$ is a new differential on Y (Such t is called a perturbation). Then*

$$\begin{aligned}\partial_\infty &= \partial + \sum_{k>0} f(t\Phi)^{k-1}t\nabla, \\ \nabla_\infty &= \nabla + \sum_{k>0} (\Phi t)^k \nabla, \\ f_\infty &= f + \sum_{k>0} f(t\Phi)^k, \\ \Phi_\infty &= \Phi + \sum_{k>0} (\Phi t)^k \Phi\end{aligned}$$

are well defined linear maps and $(X, \partial_\infty) \xrightarrow[\nabla_\infty]{f_\infty} (Y, d + t) \circ \Phi_\infty$ is a strong deformation retract.

By applying the Perturbation Lemma to Proposition 5.2 we rediscover

Theorem 5.4 [33, Théorème 1.4] *Let (TAV, d) be a chain algebra. Suppose that V is a graded module concentrated in degree greater or equal than one. Define the linear map of degree +1*

$$\begin{aligned} S : \text{TV} \otimes \text{TV} &\rightarrow \text{TV} \otimes sV \otimes \text{TV} \\ a \otimes v_1 \cdots v_n &\mapsto \sum_{i=1}^n (-1)^{|av_1 \cdots v_{i-1}|} av_1 \cdots v_{i-1} \otimes sv_i \otimes v_{i+1} \cdots v_n \end{aligned}$$

Consider the chain complex $(\text{TV} \otimes (\mathbb{k} \oplus sV) \otimes \text{TV}, D)$ where

$$D|_{\text{TV} \otimes \text{TV}} = d, \quad D|_{\text{TV} \otimes sV \otimes \text{TV}} = \tilde{d}_1 + d_2,$$

$$\tilde{d}_1(a \otimes sv \otimes b) = da \otimes sv \otimes b - S(a \otimes dv).b - (-1)^{|av|} a \otimes sv \otimes db$$

$$\text{and } d_2(a \otimes sv \otimes b) = (-1)^{|a|}(av \otimes b - a \otimes vb) \text{ for } a, b \in \text{TV}, v \in V.$$

Then $(\text{TV} \otimes (\mathbb{k} \oplus sV) \otimes \text{TV}, D)$ is a strong deformation retract of the (TAV, TAV) -bimodule $B(\text{TAV}, \text{TAV}, \text{TAV})$.

Proof. By Proposition 5.2 $(\text{TV} \otimes (\mathbb{k} \oplus sV) \otimes \text{TV}, d_1 + d_2)$ is a strong deformation retract of $B((\text{TAV}, d_1), (\text{TAV}, d_1), (\text{TAV}, d_1))$ where d_1 denotes the linear part of d . The annihilation conditions are satisfied:

$$\Phi_1(a[sv]b) = 0 \text{ and } \Phi_0 = 0, \text{ therefore } \Phi \nabla = 0.$$

The projection f is null on $\text{TV} \otimes (sT^+V)^{\otimes \geq 2} \otimes \text{TV}$ and $\Phi_0 = 0$. Therefore $f\Phi_n = 0$ for $n \in \mathbb{N}$.

Since $\Phi_{n+1}\Phi_n(a[sa_1|\cdots|sa_nv]b) = \Phi_{n+1}\Phi_n(a[sa_1|\cdots|sa_n]vb)$, by induction on wordlength $\Phi_{n+1}\Phi_n = 0$ for $n \geq 1$.

Let $n \in \mathbb{Z}$. An element $a[sa_1|\cdots|sa_n]b$ is said to have a filtration degree $-n$ if and only if the sum of the wordlengths of a, a_1, \dots, a_n and b is greater or equal than n . The filtrations are bounded below since $V = V_{\geq 1}$. The maps ∇, f, Φ, d_1 and d_2 respect wordlengths. Therefore the strong deformation retract is filtered. Define the perturbation t to be equal to the differential of $B((\text{TAV}, d), (\text{TAV}, d), (\text{TAV}, d))$ minus the differential of $B((\text{TAV}, d_1), (\text{TAV}, d_1), (\text{TAV}, d_1))$. Since $d_{\geq 2} = d - d_1$ increases wordlength by 1 at least, t is filtration lowering.

So finally we can apply the Perturbation Lemma and $(TV \otimes (\mathbb{k} \oplus sV) \otimes TV, \partial_\infty)$ is a strong deformation retract of $B((TAV, d), (TAV, d), (TAV, d))$.

The composite $t\Phi_n$ maps $TV \otimes (sT^+V)^{\otimes n} \otimes TV$ into $TV \otimes (sT^+V)^{\otimes n+1} \otimes TV$, Φ_0 is null. Therefore $f(t\Phi)^k = 0$ for $k \geq 1$. So $f = f_\infty$ (The projection is unchanged) and $D := \partial_\infty = \partial + ft\nabla = d_2 + fd_1\nabla$ where d_1 is the linear part of the the differential of $B((TAV, d), (TAV, d), (TAV, d))$. Set $\tilde{d}_1 := fd_1\nabla$.

$$\tilde{d}_1(a \otimes sv \otimes b) = da \otimes sv \otimes b - (-1)^{|a|}f_1(a \otimes s dv \otimes b) - (-1)^{|av|}a \otimes sv \otimes db.$$

$$S(a \otimes dv).b = (-1)^{|a|}f_1(a \otimes s dv \otimes b).$$

QED

Corollary 5.5 [33, Théorème 1.5] *Let (TAV, d) be a chain algebra such that $V = V_{\geq 1}$. Define the linear map of degree +1*

$$\begin{aligned} \overline{S} : TV \otimes TV &\rightarrow sV \otimes TV \\ v_1 \dots v_n \otimes a &\mapsto \sum_{i=1}^n (-1)^{|v_1 \dots v_{i-1}| |v_i \dots v_n a|} s v_i \otimes v_{i+1} \dots v_n a v_1 \dots v_{i-1} \end{aligned}$$

Consider the complex $((\mathbb{k} \oplus sV) \otimes TV, \delta)$ where

$$\delta|_{TV} = d,$$

$$\delta(sv \otimes a) = (-1)^{|a||v|}1 \otimes av - 1 \otimes va + (-1)^{|sv|}sv \otimes da - \overline{S}(dv \otimes a).$$

Then $((\mathbb{k} \oplus sV) \otimes TV, \delta)$ is a strong deformation retract of $C(TAV, d)$.

Proof. The linear maps

$$\begin{aligned} (\mathbb{k} \oplus sV) \otimes TV &\rightarrow (TV \otimes (\mathbb{k} \oplus sV) \otimes TV) \otimes_{TAV \otimes TAV^{op}} TV \\ \overline{v} \otimes b &\mapsto 1 \otimes \overline{v} \otimes 1 \otimes b \end{aligned}$$

$$\begin{aligned} (TV \otimes (\mathbb{k} \oplus sV) \otimes TV) \otimes_{TAV \otimes TAV^{op}} TV &\rightarrow (\mathbb{k} \oplus sV) \otimes TV \\ a \otimes \overline{v} \otimes a' \otimes b &\mapsto (-1)^{|a|(|\overline{v}|+|a'|+|b|)} \overline{v} \otimes a'ba \end{aligned}$$

are inverse. The strong deformation retract given by Theorem 5.4 is compatible with the structure of (TAV, TAV) -bimodule on $B(TAV; TAV; TAV)$. To obtain the Corollary, we should tensor it by $TV \otimes_{TAV \otimes TAV^{op}} -$ and then

permute TV and $(\mathbb{k} \oplus sV)$. But it is equivalent and shorter to tensor by $-\otimes_{TAV \otimes TAV^{op}} TV$ and use the previous isomorphisms. \boxed{QED}

Suppose that (TAV, ∂) is a HAH model of a path connected space X . Using the inclusion ∇_∞ and the projection $f = f_\infty$ of the strong deformation retract given by Corollary 5.5, it is now easy to transport the diagonal of $C(TAV, \partial)$, denoted $\Delta_{C(TAV, \partial)}$, to $((\mathbb{k} \oplus sV) \otimes TV, \delta)$. Define the diagonal of $((\mathbb{k} \oplus sV) \otimes TV, \delta)$ simply as the composite $(f \otimes f) \circ \Delta_{C(TAV, \partial)} \circ \nabla_\infty$. Now $((\mathbb{k} \oplus sV) \otimes TV, \delta)^\vee$ is an algebra up to homotopy whose homology is isomorphic to $HH^*(TAV, \partial)$ as graded algebras. This algebra up to homotopy is the smallest that computes in general the cohomology algebra of the free loop space $H^*(X^{S^1})$. But the formula for the diagonal of $((\mathbb{k} \oplus sV) \otimes TV, \delta)$ is very complicated: it involves in particular the formula of the inclusion ∇_∞ given by the Perturbation Lemma.

We will now limit ourself to two important cases where the HAH structure on (TAV, ∂) is simple:

- The differential ∂ is the sum $d_1 + d_2$ of only its linear part d_1 and its quadratic part d_2 . The elements of V are primitive: TAV is a primitively generated Hopf algebra. This will be the object of Section 6.
- The differential ∂ is equal to its linear part d_1 (hypothesis of Proposition 5.2). The reduced diagonal $\overline{\Delta}$ of TAV is such that $\overline{\Delta}(V) \subset V \otimes V$. This will be the object of Section 7.

6 The isomorphism between $HH^*(\Omega C)$ and $HH_*(C^\vee)$

Let C be a coaugmented DGC. Denote by \overline{C} the kernel of the counit. The *co-bar construction on C* , denoted ΩC , is the augmented DGA $(TA(s^{-1}\overline{C}), d_1 + d_2)$ where d_1 and d_2 are the unique derivations determined by

$$d_1 s^{-1}c = -s^{-1}dc \text{ and } d_2 s^{-1}c = \sum_i (-1)^{|x_i|} s^{-1}x_i \otimes s^{-1}y_i, \quad c \in \overline{C}$$

where the reduced diagonal $\overline{\Delta}c = \sum_i x_i \otimes y_i$. We follow the sign convention of [12].

Theorem 6.1 [19, Theorem A][17, Theorem II] Consider a coaugmented DGC C \mathbb{k} -free of finite type such that $C = \mathbb{k} \oplus C_{\geq 2}$. Then there is a natural isomorphism of graded modules

$$HH^*(\Omega C) \cong HH_*(C^\vee).$$

We give again the proof of Jones and McCleary since we want to check carefully the signs. We also need to explicit the isomorphism in order to transport later the algebra structure. We remark that already at the level of complexes, there is a quasi-isomorphism from $C(\Omega C)^\vee$ to $C(C^\vee)$.

Before giving the proof, we need to give the signs convention used in this paper. Let $f : V \rightarrow V'$ and $g : W \rightarrow W'$ be two linear maps then $f \otimes g : V \otimes W \rightarrow V' \otimes W'$ is the linear map given by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

Therefore if $f' : V' \rightarrow V''$ and $g' : W' \rightarrow W''$ are two other linear maps then

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|f||g'|} (f' \circ g') \otimes (f \circ g).$$

Let $\varphi : M \rightarrow N$ be a linear map. If $f \in \text{Hom}(N, \mathbb{k})$ then

$$\varphi^\vee(f) = (-1)^{|f||\varphi|} f \circ \varphi.$$

In particular, if (M, d) is a complex, the dual complex is (M^\vee, d^\vee) . Let $\Psi : N \rightarrow Q$ be another linear map. Then $(\Psi \circ \varphi)^\vee = (-1)^{|\varphi||\Psi|} \varphi^\vee \circ \Psi^\vee$. **Proof.** We apply Corollary 5.5 when the chain algebra (TAV, d) is the cobar ΩC . We obtain a strong deformation retract of the cyclic bar construction $C(\Omega C)$ of the form $(C \otimes \Omega C, \delta)$. The differential δ is given by

$$\begin{aligned} \delta a &= d_\Omega a, \quad a \in \Omega C, \\ \delta(c \otimes a) &= dc \otimes a + (-1)^{|c|} c \otimes da \\ &\quad - 1 \otimes (s^{-1}c)a - (-1)^{|x_i|} x_i \otimes (s^{-1}y_i)a \\ &\quad + (-1)^{|a||s^{-1}c|} 1 \otimes as^{-1}c + (-1)^{(|a|+|y_i|)|s^{-1}x_i|} y_i \otimes as^{-1}x_i. \end{aligned}$$

Therefore $(C \otimes \Omega C, \delta)$ is the complex $(C \otimes \text{T}s^{-1}\overline{C}, d_1 + d_2)$ where d_1 is just obtained by tensorization and $d_2 : C \otimes (s^{-1}\overline{C})^{\otimes n-1} \rightarrow C \otimes (s^{-1}\overline{C})^{\otimes n}$ is the sum of $n+1$ terms $\delta_0, \delta_1, \dots, \delta_n$ given by

$$\delta_0 = -[(C \otimes s^{-1}) \circ \Delta] \otimes (s^{-1}\overline{C})^{\otimes n-1},$$

$$\delta_i = C \otimes (s^{-1}\overline{C})^{\otimes i-1} \otimes [(s^{-1} \otimes s^{-1}) \circ \Delta \circ s] \otimes (s^{-1}\overline{C})^{\otimes n-1-i}, \quad 1 \leq i \leq n-1$$

$$\text{and } \delta_n = [C \otimes (s^{-1}\overline{C})^{\otimes n-1} \otimes s^{-1}] \circ \tau_{C, C \otimes (s^{-1}\overline{C})^{\otimes n-1}} \circ [\Delta \otimes (s^{-1}\overline{C})^{\otimes n-1}].$$

Let A denote the augmented DGA C^\vee . The differential $d_2 : A \otimes (s\overline{A})^{\otimes n} \rightarrow A \otimes (s\overline{A})^{\otimes n-1}$ is also the sum of $n+1$ terms d_0, d_1, \dots, d_n (compare to Example 3.2) given by

$$d_0 = [\mu \circ (A \otimes s^{-1})] \otimes (s\overline{A})^{\otimes n-1},$$

$$d_i = A \otimes (s\overline{A})^{\otimes i-1} \otimes [s \circ \mu \circ (s^{-1} \otimes s^{-1})] \otimes (s\overline{A})^{\otimes n-i-1}, \quad 1 \leq i \leq n-1$$

$$\text{and } d_n = -[\mu \otimes (s\overline{A})^{\otimes n-1}] \circ \tau_{A \otimes (s\overline{A})^{\otimes n-1}, A} \circ [A \otimes (s\overline{A})^{\otimes n-1} \otimes s^{-1}].$$

The isomorphism $\Theta : s(\overline{C}^\vee) \xrightarrow{\cong} (s^{-1}\overline{C})^\vee$ is such that $(s^{-1})^\vee \circ \Theta = s^{-1}$ [16, p. 276]. For any two complexes V and W , the map $\Phi : V^\vee \otimes W^\vee \rightarrow (V \otimes W)^\vee$ given by $\Phi(f \otimes g) = \mu_{\mathbb{k}} \circ (f \otimes g)$ is a morphism of complexes and is associative, commutative, natural with respect to linear maps of any degree. Therefore the composite

$$A \otimes (s\overline{A})^{\otimes n} \xrightarrow{A \otimes \Theta^{\otimes n}} C^\vee \otimes [(s^{-1}\overline{C})^\vee]^{\otimes n} \xrightarrow{\Phi} [C \otimes (s^{-1}\overline{C})^{\otimes n}]^\vee$$

commutes with d_i and δ_i^\vee for $0 \leq i \leq n$. So finally

$$\Phi \circ [A \otimes T(\Theta)] : C(A) \xrightarrow{\cong} (C \otimes \Omega C, \delta)^\vee$$

is an isomorphism of complexes. \boxed{QED}

Let V be a graded module. The tensor algebra TAV can be made into a cocommutative Hopf algebra by requiring the elements of V to be primitive [34, 0.5 (10)]. We will call the resulting diagonal, the *shuffle diagonal*. Dually the tensor coalgebra TCV equipped with the *shuffle product* is a commutative Hopf algebra. The shuffle product is defined by

$$[v_1 | \dots | v_p] \cdot [v_{p+1} | \dots | v_{p+q}] = \sum_{\sigma} \sigma \cdot [v_1 | \dots | v_{p+q}]$$

where the sum is taken over the (p, q) -shuffles σ and a permutation σ acts on $[v_1 | \dots | v_{p+q}]$ by permuting the factors with appropriate signs [16, Appendix] or [34, 0.5 (8)]. Suppose that V is \mathbb{k} -free of finite type and $V = V_{\geq 1}$. Then the map $\Phi : TC(V^\vee) \xrightarrow{\cong} (TAV)^\vee$ is an isomorphism of Hopf algebras.

Let C be a cocommutative coaugmented DGC. Then the cobar $\Omega C = (\mathrm{TA}(s^{-1}\overline{C}), d_1 + d_2)$ equipped with the shuffle diagonal is a DGH [34, 0.6 (2)]. Dually, let A be an augmented CDGA. Consider the multiplication on $A \otimes \mathrm{TC}(s\overline{A}) \otimes A$ obtained by tensorizing the multiplication of A and the shuffle product of $\mathrm{TC}(s\overline{A})$. Then the bar resolution of A , $B(A; A; A) = (A \otimes \mathrm{TC}(s\overline{A}) \otimes A, d_1 + d_2)$ is a CDGA. The cyclic bar construction $C(A) = (A \otimes \mathrm{TC}(s\overline{A}), d_1 + d_2)$ is also a CDGA. Therefore the Hochschild homology of a CDGA A , $HH_*(A)$, has a natural structure of commutative graded algebra [23, 4.2.7]. The reduced bar construction of A , $B(A) = (\mathrm{TC}(s\overline{A}), d_1 + d_2)$ is a commutative DGH [34, 0.6 (1)].

Theorem 6.2 *Under the hypothesis of Theorem 6.1, if C is cocommutative then the isomorphism $HH^*(\Omega C) \cong HH_*(C^\vee)$ is an isomorphism of commutative graded algebras.*

Property 6.3 Let K be a graded Hopf algebra. Consider $\mathrm{Ker}\overline{\Delta}$ the primitive elements of K and the graded coalgebra $K \otimes \mathrm{TC}(s\mathrm{Ker}\overline{\Delta}) \otimes K$ obtained by tensorization. Then the canonical map $K \otimes \mathrm{TC}(s\mathrm{Ker}\overline{\Delta}) \otimes K \rightarrow B(K; K; K)$ is a morphism of graded coalgebras.

Proof of Theorem 6.2

When restricted to conilpotent coaugmented DGC's, the cobar construction Ω is a left adjoint functor to the bar construction B [12, Proposition 2.11]. By Formula 2.1, the adjunction map $\sigma_C : C \xrightarrow{\cong} B\Omega C$ is given by

$$\sigma_C(c) = \sum_{i=0}^{+\infty} \sum [ss^{-1}c_1 | \cdots | ss^{-1}c_{i+1}], \quad c \in \overline{C}$$

where the iterated reduced diagonal $\overline{\Delta}^i c = \sum c_1 \otimes \cdots \otimes c_{i+1}$. We consider now the inclusion map ∇_∞ of the strong deformation retract given by Theorem 5.4 when the chain algebra (TAV, d) is the cobar ΩC . A simple computation shows that ∇_∞ is $\Omega C \otimes \sigma_C \otimes \Omega C$, the tensor product of the identity maps and the adjunction map. Now σ_C is a morphism of coalgebras, $\mathrm{Im}\sigma_C \subset \mathrm{TC}(ss^{-1}\overline{C})$ and $s^{-1}\overline{C} \subset \mathrm{Ker}\overline{\Delta}$. Therefore by Property 6.3, the coalgebra $\Omega C \otimes C \otimes \Omega C$ obtained by tensorization is a sub DGC of $B(\Omega C; \Omega C; \Omega C)$. After tensorizing by $- \otimes_{\Omega C \otimes \Omega C^{op}} \Omega C$ and dualizing, we obtain the natural DGA quasi-isomorphism

$$C(C^\vee) \xrightarrow{\cong} (C \otimes \Omega C, \delta)^\vee \xleftarrow{\cong} C(\Omega C)^\vee.$$

QED

7 The free loop space on a suspension

In this section, we show how to compute the cohomology algebra of the free loop space on any suspension, $H^*((\Sigma X)^{S^1})$. And even better, from the DGA $S^*(X)$ or any DGA weakly equivalent, we construct an DGA weakly equivalent to the DGA $S^*((\Sigma X)^{S^1})$.

First we introduce some terminology. Let C be a coaugmented DGC. The composite $C \xrightarrow{\Delta_C} C \otimes C \hookrightarrow \mathrm{TA}\overline{C} \otimes \mathrm{TA}\overline{C}$ extends to a unique morphism of augmented DGA's

$$\Delta_{\mathrm{TA}\overline{C}} : \mathrm{TA}\overline{C} \rightarrow \mathrm{TA}\overline{C} \otimes \mathrm{TA}\overline{C}.$$

This DGH structure on the tensor algebra $\mathrm{TA}\overline{C}$ is called the *Hopf algebra structure obtained by tensorization of the coalgebra C* . Dually, let A be an augmented DGA. The composite $\mathrm{TC}\overline{A} \otimes \mathrm{TC}\overline{A} \rightarrow A \otimes A \xrightarrow{\mu_A} A$ lifts to a unique morphism of coaugmented DGC's

$$\mu_{\mathrm{TC}\overline{A}} : \mathrm{TC}\overline{A} \otimes \mathrm{TC}\overline{A} \rightarrow \mathrm{TC}\overline{A}.$$

This DGH structure on the tensor coalgebra $\mathrm{TC}\overline{A}$ is called the *Hopf algebra structure obtained by tensorization of the algebra A* . Using formula 2.1, we see that the product $\mu_{\mathrm{TC}\overline{A}}$ of two elements $[a_1 | \cdots | a_p]$ and $[b_1 | \cdots | b_q]$ admits the following description:

A sequence $\sigma = ((0, 0) = (x_0, y_0), (x_1, y_1), \dots, (x_n, y_n) = (p, q))$ defined by

$$(x_i, y_i) = \begin{cases} (x_{i-1} + 1, y_{i-1}) & \text{or} \\ (x_{i-1}, y_{i-1} + 1) & \text{or} \\ (x_{i-1} + 1, y_{i-1} + 1), \end{cases}$$

is called a *step by step path* of length n from $(0, 0)$ to (p, q) . To any step by step path σ of length n , we associate $c_\sigma = [c_1 | \cdots | c_n] \in \overline{A}^{\otimes n}$ by the rule

$$c_i = \begin{cases} a_{x_i} & \text{if } (x_i, y_i) = (x_{i-1} + 1, y_{i-1}), i^{th} \text{ step is toward right,} \\ b_{y_i} & \text{if } (x_i, y_i) = (x_{i-1}, y_{i-1} + 1), i^{th} \text{ step is toward up,} \\ \mu_A(a_{x_i} \otimes b_{y_i}) & \text{if } (x_i, y_i) = (x_{i-1} + 1, y_{i-1} + 1), i^{th} \text{ step in diagonal.} \end{cases}$$

Then a straightforward computation establishes

$$\mu_{\mathrm{TC}\overline{A}}([a_1|\cdots|a_p]\otimes[b_1|\cdots|b_q])=\sum_{\sigma}\pm c_{\sigma} \quad (7.1)$$

where the sum is taken over all the step by step paths σ from $(0,0)$ to (p,q) and where \pm is the sign obtained with Koszul rule by mixing the a_1,\cdots,a_p and the b_1,\cdots,b_q . In particular, when the product of A is trivial, the product $\mu_{\mathrm{TC}\overline{A}}$ is the shuffle product considered page 21.

Suppose that the DGC C is \mathbb{k} -free of finite type and such that $C = \mathbb{k} \oplus C_{\geq 1}$. Then the map

$$\Phi : \mathrm{TC}(\overline{C^\vee}) \xrightarrow{\cong} (\mathrm{TA}\overline{C})^\vee$$

is a DGH isomorphism.

The starting observation of this section is the following consequence of Bott-Samelson Theorem (see [28, 7.1] for details).

Lemma 7.2 *Let X be a path connected space.*

i) *Consider the Hopf algebra structure on $\mathrm{TA}\overline{S}_*(X)$ obtained by tensorization of the coalgebra $S_*(X)$. Then there is a natural DGH quasi-isomorphism*

$$\mathrm{TA}\overline{S}_*(X) \xrightarrow{\cong} S_*(\Omega\Sigma X).$$

ii) *Suppose that $H_*(X)$ is \mathbb{k} -free. Consider the Hopf algebra structure on $\mathrm{TA}H_+(X)$ obtained by tensorization on the coalgebra $H_*(X)$. Then there is a HAH quasi-isomorphism*

$$\Theta_X : \mathrm{TA}H_+(X) \xrightarrow{\cong} S_*(\Omega\Sigma X).$$

We can choose Θ_X to be natural in homology and so natural after passing to homotopy of algebras (Property 4.5 i)).

Definition 7.3 [29, 2.2.11] Let R be a graded algebra. Let M be a (R, R) -bimodule. The graded module $R \oplus M$, product of R and of M equipped with the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 \cdot m_2 + m_1 \cdot r_2)$$

is a graded algebra called the *trivial extension of R by M* .

Lemma 7.4 *Let C be a chain coalgebra \mathbb{k} -free of finite type such that $C = \mathbb{k} \oplus C_{\geq 1}$. Denote by A the cochain algebra dual of C . Consider the Hopf algebra structures on $\mathrm{TA}\overline{C}$ and on $\mathrm{TC}\overline{A}$ obtained by tensorization of the coalgebra C and the algebra A . Define a structure of $(\mathrm{TC}\overline{A}, \mathrm{TC}\overline{A})$ -bimodule on $s^{-1}\overline{A} \otimes \mathrm{T}\overline{A}$ by*

$$(s^{-1}a \otimes m) \cdot a_1 \dots a_n = -s^{-1}a \otimes \mu_{\mathrm{TC}\overline{A}}(m \otimes a_1 \dots a_n) \\ - (-1)^{|a_n|(|m|+|a_1 \dots a_{n-1}|)} s^{-1}\mu_A(a \otimes a_n) \otimes \mu_{\mathrm{TC}\overline{A}}(m \otimes a_1 \dots a_{n-1})$$

and

$$a_1 \dots a_n \cdot (s^{-1}a \otimes m) = -(-1)^{|s^{-1}a||a_1 \dots a_n|} s^{-1}a \otimes \mu_{\mathrm{TC}\overline{A}}(a_1 \dots a_n \otimes m) \\ - (-1)^{|a_1|+|s^{-1}a||a_2 \dots a_n|} s^{-1}\mu_A(a_1 \otimes a) \otimes \mu_{\mathrm{TC}\overline{A}}(a_2 \dots a_n \otimes m)$$

for $a \in \overline{A}$, $m \in \mathrm{T}\overline{A}$ and $a_1 \dots a_n \in \mathrm{TC}\overline{A}$. Consider the cochain complex $(\mathbb{k} \oplus s^{-1}\overline{A})$ equipped with the trivial product. Then the cyclic bar construction on $(\mathbb{k} \oplus s^{-1}\overline{A})$, $C(\mathbb{k} \oplus s^{-1}\overline{A}) = \mathrm{TC}\overline{A} \oplus (s^{-1}\overline{A} \otimes \mathrm{T}\overline{A})$, equipped with the product of the trivial extension of $\mathrm{TC}\overline{A}$ by $s^{-1}\overline{A} \otimes \mathrm{T}\overline{A}$ is a cochain algebra equipped with a natural DGA quasi-isomorphism

$$C(\mathbb{k} \oplus s^{-1}\overline{A}) \xrightarrow{\simeq} C(\mathrm{TA}\overline{C})^\vee.$$

Property 7.5 Let K be a graded Hopf algebra. The diagonal of the coalgebra $B(K; K; K)$ restricted to $K \otimes (\mathbb{k} \oplus s\overline{K}) \otimes K$ is the (K, K) -linear map given by

$$\Delta[sx] = [sx] \otimes [] + [sy] \otimes z[] + (-1)^{|y|} []y \otimes [sz] + [] \otimes [sx], x \in \overline{K}$$

where the reduced diagonal $\overline{\Delta}x = \sum y \otimes z$.

Proof of Lemma 7.4 The tensor algebra $\mathrm{TA}\overline{C}$ is equal as DGA to the cobar on the DGC $\mathbb{k} \oplus s\overline{C}$ with trivial coproduct. Therefore by Theorem 6.1, we get immediatly a natural quasi-isomorphism of cochain complexes

$$C(\mathbb{k} \oplus s^{-1}\overline{A}) \xrightarrow{\simeq} C(\mathrm{TA}\overline{C})^\vee.$$

But we have to remember how this morphism decomposes in order to transport the multiplication from $C(\mathrm{TA}\overline{C})^\vee$ to $C(\mathbb{k} \oplus s^{-1}\overline{A})$.

Since the differential on $\mathrm{TA}\overline{C}$ is only linear, by Proposition 5.2, the canonical inclusion

$$\mathrm{T}\overline{C} \otimes (\mathbb{k} \oplus s\overline{C}) \otimes \mathrm{T}\overline{C} \xrightarrow{\sim} \mathrm{B}(\mathrm{TA}\overline{C}; \mathrm{TA}\overline{C}; \mathrm{TA}\overline{C})$$

is a quasi-isomorphism of complexes. Since the reduced diagonal of $\mathrm{TA}\overline{C}$, $\overline{\Delta}_{\mathrm{TA}\overline{C}}$ embeds \overline{C} into $\overline{C} \otimes \overline{C}$, by Property 7.5, $\mathrm{T}\overline{C} \otimes (\mathbb{k} \oplus s\overline{C}) \otimes \mathrm{T}\overline{C}$ is a subcoalgebra of $\mathrm{B}(\mathrm{TA}\overline{C}; \mathrm{TA}\overline{C}; \mathrm{TA}\overline{C})$. By tensorizing by $-\otimes_{\mathrm{TA}\overline{C} \otimes \mathrm{TA}\overline{C}} \mathrm{TA}\overline{C}$, we obtain the DGC $(\mathbb{k} \oplus s\overline{C}) \otimes \mathrm{T}\overline{C}$ with differential given by Corollary 5.5 and diagonal given by

$$\begin{aligned} \Delta(sx \otimes c) = & sx \otimes c' \otimes 1 \otimes c'' + (-1)^{|c||z|} sy \otimes c' \otimes 1 \otimes c''z \\ & + (-1)^{|y|+|c'||sz|} 1 \otimes yc' \otimes sz \otimes c'' + (-1)^{|c'||sx|} 1 \otimes c' \otimes sx \otimes c'' \end{aligned}$$

for $x \in \overline{C}$, $c \in \mathrm{T}\overline{C}$ and where the reduced diagonal $\overline{\Delta}x = \sum y \otimes z$ and the unreduced diagonal $\Delta c = \sum c' \otimes c''$. The canonical inclusion

$$(\mathbb{k} \oplus s\overline{C}) \otimes \mathrm{T}\overline{C} \xrightarrow{\sim} \mathrm{C}(\mathrm{TA}\overline{C})$$

is a DGC quasi-isomorphism.

In order to dualize (for details, review the proof of Theorem 6.1), we see that the diagonal on $(\mathbb{k} \oplus s\overline{C}) \otimes \mathrm{T}\overline{C}$ is the sum of three terms,

$$\Delta_{\mathrm{TA}\overline{C}} : \mathrm{T}\overline{C} \rightarrow \mathrm{T}\overline{C} \otimes \mathrm{T}\overline{C},$$

$$\Delta_1 : s\overline{C} \otimes \mathrm{T}\overline{C} \rightarrow s\overline{C} \otimes \mathrm{T}\overline{C} \otimes \mathrm{T}\overline{C}$$

and

$$\Delta_2 : s\overline{C} \otimes \mathrm{T}\overline{C} \rightarrow \mathrm{T}\overline{C} \otimes s\overline{C} \otimes \mathrm{T}\overline{C}.$$

The first term $\Delta_{\mathrm{TA}\overline{C}}$ is just the diagonal of $\mathrm{TA}\overline{C}$. The second term Δ_1 is the composite

$$(s \otimes \mathrm{T}\overline{C} \otimes \mu_{\mathrm{TA}\overline{C}}) \circ (C \otimes \mathrm{T}\overline{C} \otimes \mathrm{T}\overline{C} \otimes i) \circ (C \otimes \tau_{C, \mathrm{T}\overline{C} \otimes \mathrm{T}\overline{C}}) \circ (\Delta_C \otimes \Delta_{\mathrm{TA}\overline{C}}) \circ (s^{-1} \otimes \mathrm{T}\overline{C})$$

where i denotes the inclusion $C \hookrightarrow \mathrm{T}\overline{C}$. The third term Δ_2 is the composite

$$(\mu_{\mathrm{TA}\overline{C}} \otimes s \otimes \mathrm{T}\overline{C}) \circ (i \otimes \tau_{C, \mathrm{T}\overline{C}} \otimes \mathrm{T}\overline{C}) \circ (\Delta_C \otimes \Delta_{\mathrm{TA}\overline{C}}) \circ (s^{-1} \otimes \mathrm{T}\overline{C}).$$

Therefore the product on $(\mathbb{k} \oplus s^{-1}\overline{A}) \otimes \mathrm{T}\overline{A}$ is the sum of three terms: the product $\mu_{\mathrm{TC}\overline{A}}$ of $\mathrm{TC}\overline{A}$, the dual of $\Delta_1 : s^{-1}\overline{A} \otimes \mathrm{T}\overline{A} \otimes \mathrm{T}\overline{A} \rightarrow s^{-1}\overline{A} \otimes \mathrm{T}\overline{A}$ and the dual of $\Delta_2 : \mathrm{T}\overline{A} \otimes s^{-1}\overline{A} \otimes \mathrm{T}\overline{A} \rightarrow s^{-1}\overline{A} \otimes \mathrm{T}\overline{A}$.

Explicitly the product is given by

$$\begin{aligned}
(1 \otimes m).(1 \otimes m') &= 1 \otimes \mu_{\text{TC}\overline{A}}(m \otimes m'), \\
(s^{-1}a \otimes m).(1 \otimes a_1 \dots a_n) &= -s^{-1}a \otimes \mu_{\text{TC}\overline{A}}(m \otimes a_1 \dots a_n) \\
&\quad - \pm s^{-1}\mu_A(a \otimes a_n) \otimes \mu_{\text{TC}\overline{A}}(m \otimes a_1 \dots a_{n-1}), \\
(1 \otimes a_1 \dots a_n).(s^{-1}a \otimes m) &= -\pm s^{-1}a \otimes \mu_{\text{TC}\overline{A}}(a_1 \dots a_n \otimes m) \\
&\quad - \pm s^{-1}\mu_A(a_1 \otimes a) \otimes \mu_{\text{TC}\overline{A}}(a_2 \dots a_n \otimes m)
\end{aligned}$$

and

$$(s^{-1}a \otimes m).(s^{-1}a' \otimes m') = 0.$$

for $a, a' \in \overline{A}$, m, m' and $a_1 \dots a_n \in \text{TC}\overline{A}$ and where \pm are the signs obtained exactly by the Koszul sign convention. QED

Theorem 7.6 *Let X be a path connected space. If $S_*(X)$ is weakly equivalent as \mathbb{k} -free chain coalgebra to a chain coalgebra C \mathbb{k} -free of finite type such that $C = \mathbb{k} \oplus C_{\geq 1}$. Then the singular cochains on the free loop spaces on the suspension of X , $S^*((\Sigma X)^{S^1})$ is weakly equivalent as cochain algebras to the cyclic bar construction $C(\mathbb{k} \oplus s^{-1}\overline{C}^\vee)$ equipped with the product of the trivial extension given by Lemma 7.4.*

Remark 7.7 If X is a finite simply connected CW-complex then X satisfies the hypothesis of Theorem 7.6. Indeed, the Adams-Hilton model of X denoted $\mathcal{A}(X)$ is a free chain algebra (TAV, ∂) equipped with a quasi-isomorphism of augmented chain algebras $(\text{TAV}, \partial) \xrightarrow{\cong} S_*(\Omega X)$ and such that the complex of indecomposables (V, d_1) is the desuspension of the reduced cellular chain complex of X . Now the bar construction $BS_*(\Omega X)$ is weakly equivalent as \mathbb{k} -free chain coalgebras to $S_*(X)$. Therefore we can take $C = B\mathcal{A}(X)$.

Example 7.8 S^d , $d \geq 1$. If $d \geq 2$ using Remark 7.7, $S_*(S^d)$ is weakly DGC equivalent to $B\Omega H_*(S^d)$, therefore to $H_*(S^d)$. By [13, 7.3], there is a DGH quasi-isomorphism $H_*(S^1) \xrightarrow{\cong} S_*(S^1)$. So by Theorem 7.6, as DGA

$$S^*((S^{d+1})^{S^1}) \sim CH^*(S^{d+1}) = E(s^{-1}v) \otimes \text{TC}v, d_2$$

where v is an element of degree d . If d is even in \mathbb{k} , as DGA

$$S^* \left((S^{d+1})^{S^1} \right) \sim E(s^{-1}v) \otimes \Gamma v, 0$$

and

$$H^* \left((S^{d+1})^{S^1} \right) \cong H^*(S^{d+1}) \otimes H^*(\Omega S^{d+1})$$

as graded algebras. We suppose now that d is odd. By dualization, $\mathrm{TC}v \cong Ev \otimes \mathrm{TC}(v^2)$ as graded algebras (James-Toda). So as cochain algebras

$$S^* \left((S^{d+1})^{S^1} \right) \sim E(s^{-1}v) \otimes Ev \otimes \Gamma(v^2), d_2 \gamma_k(v^2) = 2(s^{-1}v)v\gamma_{k-1}(v^2), k \geq 1.$$

Therefore, over any commutative ring \mathbb{k} , the graded algebra $H^* \left((\Sigma S^d)^{S^1} \right)$ is the module

$$\mathbb{k} \oplus ({}_2\mathbb{k})\Gamma^+(v^2) \oplus \mathbb{k}v.\Gamma(v^2) \oplus \mathbb{k}(s^{-1}v).\Gamma(v^2) \oplus \left(\frac{\mathbb{k}}{2\mathbb{k}}\right)v.(s^{-1}v).\Gamma(v^2)$$

equipped with the obvious products. In particular, if $\frac{1}{2} \in \mathbb{k}$, all the products are trivial.

Example 7.9 Comparaison of \mathbb{CP}^d and $S^2 \vee \dots \vee S^{2d}$, $d \geq 1$. The Adams-Hilton model of \mathbb{CP}^d is $\Omega H_*(\mathbb{CP}^d)$. Therefore (Remark 7.7), $S_*(\mathbb{CP}^d)$ is weakly DGC equivalent to $H_*(\mathbb{CP}^d)$. So by Theorem 7.6, as cochain algebras

$$S^* \left((\Sigma \mathbb{CP}^d)^{S^1} \right) \sim C(H^*(\Sigma \mathbb{CP}^d)).$$

Similarly as cochain algebras

$$S^* \left((S^3 \vee \dots \vee S^{2d+1})^{S^1} \right) \sim C(H^*(S^3 \vee \dots \vee S^{2d+1})).$$

The cochain algebras $C(H^*(\Sigma \mathbb{CP}^d))$ and $C(H^*(S^3 \vee \dots \vee S^{2d+1}))$ have the same underlying cochain complex. But the non commutative product on $C(H^*(\Sigma \mathbb{CP}^d))$ given by Lemma 7.4 and using formula 7.1, is far more complicated than the shuffle product on $C(H^*(S^3 \vee \dots \vee S^{2d+1}))$.

Proof of Theorem 7.6 By Lemma 7.2 i) and Theorem 3.1 (same proof as Theorem 4.6), there is a natural DGC quasi-isomorphism

$$C(\mathrm{TAS}_*(X)) \xrightarrow{\cong} S_*((\Sigma X)^{S^1}).$$

Since the cyclic bar construction (Property 4.1), dualizing and tensorization preserve quasi-isomorphisms between \mathbb{k} -free chain complexes, $C(\text{TA}\overline{S}_*(X))^\vee$ is weakly equivalent as cochain algebras to $C(\text{TA}\overline{C})^\vee$. By Lemma 7.4, there is a DGA quasi-isomorphism

$$C(\text{TA}\overline{C})^\vee \xrightarrow{\sim} C(\mathbb{k} \oplus s^{-1}\overline{C}^\vee).$$

Therefore $C(\mathbb{k} \oplus s^{-1}\overline{C}^\vee)$ is a weakly DGA equivalent to $S^*((\Sigma X)^{S^1})$. QED

We give now the module structure of the Hochschild homology of a graded algebra with trivial product. Let V be a graded module. The circular permutation toward right τ acts on $T^n V$ by

$$\tau.[v_1 | \dots | v_n] = (-1)^{|v_n||v_1 \dots v_{n-1}|} [v_n | v_1 | \dots | v_{n-1}].$$

Define the *invariants* by

$$T^n V^\tau = \{x \in T^n V, \tau.x = x\}, TV^\tau = \bigoplus_{n=0}^\infty T^n V^\tau$$

and the *coinvariants* by

$$T^n V_\tau = \frac{T^n V}{\{x - \tau.x, x \in T^n V\}}, TV_\tau = \bigoplus_{n=0}^\infty T^n V_\tau$$

Consider the graded algebra $(\mathbb{k} \oplus s^{-1}V)$ with trivial product. The Hochschild homology of $(\mathbb{k} \oplus s^{-1}V)$ is the sum of the invariants and of the coinvariants of positive length desuspended:

$$HH_*(\mathbb{k} \oplus s^{-1}V) = TV^\tau \oplus (s^{-1} \otimes 1)\overline{TV_\tau}.$$

Assume now that V is \mathbb{k} -free with basis β . Then the set of words of length n on β , denoted β_n , is a basis for $T^n V = V^{\otimes n}$. The circular permutation τ acts obviously on β_n without sign:

$$\tau * v_1 \dots v_n = v_n v_1 \dots v_{n-1}.$$

Consider the quotient of the set β_n by the cyclic group generated by τ , $\langle \tau \rangle$. We denote this quotient set $\langle \tau \rangle \backslash \beta_n$. For any word $v_1 \dots v_n$ denote by k the smallest integer such that $\tau^k * v_1 \dots v_n = v_1 \dots v_n$. Of course the element of $V^{\otimes n}$, $\tau^k.[v_1 | \dots | v_n]$, is either $+[v_1 | \dots | v_n]$ or $-[v_1 | \dots | v_n]$. For any $v_1 \dots v_n$ in β_n , denote by $\text{sym}(v_1 \dots v_n)$ the element of $V^{\otimes n}$:

$$\sum_{i=0}^{k-1} \tau^i.[v_1 | \dots | v_n].$$

As modules, $T^n V^\tau$ is the direct sum

$$\bigoplus_{\overline{v_1 \dots v_n} \in \langle \tau \rangle \setminus \beta_n} \begin{cases} \mathbb{k}sym(v_1 \dots v_n) & \text{if } \tau^k \cdot [v_1 | \dots | v_n] = +[v_1 | \dots | v_n], \\ {}_2\mathbb{k}sym(v_1 \dots v_n) & \text{if } \tau^k \cdot [v_1 | \dots | v_n] = -[v_1 | \dots | v_n] \end{cases}$$

and $T^n V_\tau$ is

$$\bigoplus_{\overline{v_1 \dots v_n} \in \langle \tau \rangle \setminus \beta_n} \begin{cases} \mathbb{k}\overline{v_1 \dots v_n} & \text{if } \tau^k \cdot [v_1 | \dots | v_n] = +[v_1 | \dots | v_n], \\ \frac{\mathbb{k}}{2\mathbb{k}}\overline{v_1 \dots v_n} & \text{if } \tau^k \cdot [v_1 | \dots | v_n] = -[v_1 | \dots | v_n] \end{cases}$$

In particular, if V is concentrated in even degree or $2 = 0$ in \mathbb{k} then $HH_*(\mathbb{k} \oplus s^{-1}V)$ is \mathbb{k} -free.

We suppose now that V is \mathbb{k} -free of finite type. Using Roos direct calculation of the dimension of $T^n V^\tau$ when $\mathbb{k} = \mathbb{Q}$ [30, p. 179-80], we see (Compare [29, Theorems 1.2.1 and 1.2.2]) that the cardinal of

$$\{\overline{v_1 \dots v_n} \in \langle \tau \rangle \setminus \beta_n \text{ such that } \tau^k \cdot [v_1 | \dots | v_n] = +[v_1 | \dots | v_n] \text{ in } V \otimes_{\mathbb{k}} \dots \otimes_{\mathbb{k}} V\}$$

is given, when $1 \neq -1$ in \mathbb{k} , by

$$\frac{1}{n} \sum_{i=1}^n \sum_{v_1 \dots v_d \in \beta_d} \varepsilon(\tau^i, [v_1 | \dots | v_d | v_1 | \dots | v_d | \dots | v_1 | \dots | v_d]).$$

Here d is the greatest common divisor of i and n . And the integer

$$\varepsilon(\tau^i, [v_1 | \dots | v_d | v_1 | \dots | v_d | \dots | v_1 | \dots | v_d])$$

is the sign given by the Koszul rule derived from the action of the permutation τ^i on the element of length n , $[v_1 | \dots | v_d | v_1 | \dots | v_d | \dots | v_1 | \dots | v_d]$. In particular, by supposing that V is concentrated in even degree, we see that the cardinal of $\langle \tau \rangle \setminus \beta_n$ is

$$\frac{1}{n} \sum_{i=1}^n (\dim V)^d.$$

Let X be a path connected space such that $H_*(X)$ is \mathbb{k} -free of finite type. The Hopf algebra on $TCH^+(X)$ obtained by tensorization of the algebra $H^*(X)$ is naturally isomorphic as Hopf algebras to the loop space cohomology $H^*(\Omega \Sigma X)$. The Hochschild homology of $H^*(\Sigma X)$, $HH_*(\mathbb{k} \oplus s^{-1}H^+(X)) = TH^+(X)^\tau \oplus (s^{-1} \otimes 1) \overline{TH^+(X)}_\tau$ is naturally isomorphic as graded modules to the free loop space cohomology $H^*((\Sigma X)^{S^1})$. This isomorphism of modules is in fact an isomorphism of algebras:

Theorem 7.10 *Assume the above hypothesis. The invariants of $\mathrm{TH}^+(X)$ form a graded subalgebra, denoted $\mathrm{TCH}^+(X)^\tau$, of the loop space cohomology $H^*(\Omega\Sigma X)$. Consider the $(\mathrm{TCH}^+(X)^\tau, \mathrm{TCH}^+(X)^\tau)$ -bimodule structure on $(s^{-1} \otimes 1)\overline{\mathrm{TH}^+(X)}_\tau$ induced by the structure of $(\mathrm{TCH}^+(X), \mathrm{TCH}^+(X))$ -bimodule defined on $s^{-1}H^+(X) \otimes \mathrm{TH}^+(X)$ in Lemma 7.4. Then the associated trivial extension of $\mathrm{TCH}^+(X)^\tau$ by $(s^{-1} \otimes 1)\overline{\mathrm{TH}^+(X)}_\tau$ is naturally isomorphic as graded algebras to the free loop space cohomology of the suspension of X , $H^*((\Sigma X)^{S^1})$.*

Proof. Using Lemma 7.2 ii), Theorem 4.6 and Lemma 7.4 with $C = H_*(X)$, we obtain that the cyclic bar construction on $H^*(\Sigma X)$, $\mathrm{C}(\mathbb{k} \oplus s^{-1}H^+(X))$ equipped with the product of the trivial extension given by Lemma 7.4 for $A = H^*(X)$ has the same cohomology algebra as $H^*((\Sigma X)^{S^1})$. \square

Remark 7.11 Theorem 7.10 claims that the algebra $H^*((\Sigma X)^{S^1})$ depends functorially of the algebra $H^*(X)$. But it is useful to remember that $H^*((\Sigma X)^{S^1})$ depends functorially of the Hopf algebra structure of the loop space homology $H_*(\Omega\Sigma X) = \mathrm{TAH}_+(X)$.

For example, if we return to Example 7.9, we obtain the weak equivalences of cochain algebras

$$S^*((\Sigma\mathbb{CP}^d)^{S^1}) \sim \mathrm{C}(\mathrm{TAH}_+(\mathbb{CP}^d))^\vee \sim \mathrm{CH}^*(\Sigma\mathbb{CP}^d)$$

and

$$S^*((S^3 \vee \dots \vee S^{2d+1})^{S^1}) \sim \mathrm{C}(\mathrm{TAH}_+(S^2 \vee \dots \vee S^{2d}))^\vee \sim \mathrm{CH}^*(S^3 \vee \dots \vee S^{2d+1}).$$

If $\frac{1}{d!} \in \mathbb{k}$ then

$$\mathrm{TAH}_+(\mathbb{CP}^d) \cong \mathrm{TAH}_+(S^2 \vee \dots \vee S^{2d})$$

as graded Hopf algebras. We have the isomorphism of cochain algebras

$$\mathrm{C}(\mathrm{TAH}_+(\mathbb{CP}^d))^\vee \cong \mathrm{C}(\mathrm{TAH}_+(S^2 \vee \dots \vee S^{2d}))^\vee.$$

So finally, when $\frac{1}{d!} \in \mathbb{k}$, we have the isomorphism of graded Hopf algebras

$$H^*(\Omega\Sigma\mathbb{CP}^d) \cong H^*(\Omega(S^3 \vee \dots \vee S^{2d+1}))$$

and the isomorphism of graded algebras

$$H^* \left((\Sigma \mathbb{CP}^d)^{S^1} \right) \cong H^* \left((S^3 \vee \dots \vee S^{2d+1})^{S^1} \right).$$

The converses can be proven easily. Denote by x_2 the generator of $H^2(\Omega \Sigma \mathbb{CP}^d) \cong H^2(\Omega(S^3 \vee \dots \vee S^{2d+1}))$. In $H^*(\Omega \Sigma \mathbb{CP}^d)$, $x_2^d \neq 0$. If $d! = 0$ in \mathbb{k} , $x_2^d = 0$ in $H^*(\Omega(S^3 \vee \dots \vee S^{2d+1}))$. Therefore, when $d! = 0$ in \mathbb{k} , there is no isomorphism of graded algebras between $H^*(\Omega \Sigma \mathbb{CP}^d)$ and $H^*(\Omega(S^3 \vee \dots \vee S^{2d+1}))$. For any space X such that $H_*(X; \mathbb{Z})$ is \mathbb{Z} -free of finite type,

$$H^*(X; \mathbb{k}) \cong H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{k} \text{ and so } H^* \left(X; \frac{\mathbb{k}}{d! \mathbb{k}} \right) \cong H^*(X; \mathbb{k}) \otimes_{\mathbb{k}} \frac{\mathbb{k}}{d! \mathbb{k}}$$

as graded algebras. So we have the implications:

$$\begin{aligned} H^*(\Omega \Sigma \mathbb{CP}^d; \mathbb{k}) &\cong H^*(\Omega(S^3 \vee \dots \vee S^{2d+1}); \mathbb{k}) \text{ as graded algebras} \\ \Rightarrow H^*(\Omega \Sigma \mathbb{CP}^d; \frac{\mathbb{k}}{d! \mathbb{k}}) &\cong H^* \left(\Omega(S^3 \vee \dots \vee S^{2d+1}); \frac{\mathbb{k}}{d! \mathbb{k}} \right) \text{ as graded algebras} \\ &\Rightarrow \frac{\mathbb{k}}{d! \mathbb{k}} \text{ is the null ring} \Rightarrow \frac{1}{d!} \in \mathbb{k}. \end{aligned}$$

We prove now that if $d!$ has no inverse in \mathbb{k} , there is no isomorphism of graded algebras between $H^* \left((\Sigma \mathbb{CP}^d)^{S^1} \right)$ and $H^* \left((S^3 \vee \dots \vee S^{2d+1})^{S^1} \right)$. We have the sequence of isomorphisms of modules [23, 5.3.10]

$$H_* \left((\Sigma \mathbb{CP}^d)^{S^1} \right) \cong HH_*(\text{TA}(x_2, \dots, x_{2d})) \cong T(x_2, \dots, x_{2d})_{\tau} \oplus (s \otimes 1) \overline{T(x_2, \dots, x_{2d})^{\tau}}.$$

Thus $H_* \left((\Sigma \mathbb{CP}^d)^{S^1}; \mathbb{Z} \right)$ is \mathbb{Z} -free of finite type. So as for the loop spaces, the proof for the free loop spaces reduces to the case where $d! = 0$ in \mathbb{k} . For $X = \Sigma \mathbb{CP}^d$ or $S^2 \vee \dots \vee S^{2d}$, by Serre spectral sequence, the inclusion $\Omega X \hookrightarrow X^{S^1}$ induces in cohomology an isomorphism in degree 2 and an monomorphism in even degree. Therefore if $d! = 0$ in \mathbb{k} , $x_2^d \neq 0$ in $H^* \left((\Sigma \mathbb{CP}^d)^{S^1} \right)$ whereas $x_2^d = 0$ in $H^* \left((S^3 \vee \dots \vee S^{2d+1})^{S^1} \right)$.

It is worth noting the following particular case of Theorem 7.10. The Hochschild homology of $H^*(\Sigma X)$, $HH_*(H^*(\Sigma X))$, has a natural structure of commutative graded algebra since $H^*(\Sigma X)$ is a CDGA.

Corollary 7.12 *Let X be a path connected space such that $H_*(X)$ is \mathbb{k} -free of finite type. If the cup product on $H^*(X)$ is trivial, then $H^*((\Sigma X)^{S^1})$ is naturally isomorphic as graded algebras to $HH_*(H^*(\Sigma X))$.*

This Corollary of Theorem 7.10 is proved more easily by applying just Theorem 4.6, Lemma 7.2 ii) and Theorem 6.2.

8 The Hochschild homology of a commutative algebra

If a HAH model of a path connected pointed space X is the cobar construction on a cocommutative chain coalgebra C \mathbb{k} -free of finite type such that $C = \mathbb{k} \oplus C_{\geq 2}$, by Theorems 4.6 and 6.2, the free loop space cohomology of X , $H^*(X^{S^1})$, is isomorphic as graded algebras to the Hochschild homology of the CDGA C^\vee . In this section, we give various examples of such a space X .

Denote by A the cochain algebra C^\vee . We suppose now that A is strictly commutative (i. e. $a^2 = 0$ if $a \in A_{\text{odd}}$) and that \overline{A} is \mathbb{k} -semifree. We start by giving a method as general as possible to compute the Hochschild homology of A .

Let V be a graded module. The free strictly commutative graded algebra on V is denoted ΛV . A *decomposable Sullivan Model* of A is a cochain algebra of the form $(\Lambda V, d)$ where $V = \{V^i\}_{i \geq 2}$ is \mathbb{k} -free of finite type and $d(V) \subset \Lambda^{\geq 2} V$, equipped with a quasi-isomorphism of cochains algebras $(\Lambda V, d) \xrightarrow{\sim} A$. If \mathbb{k} is a principal ideal domain, by Theorem 7.1 of [16], A admits a minimal Sullivan model. When \mathbb{k} is a field, minimal Sullivan models are the decomposable ones [16, Remark 7.3 i)].

Anyway, suppose now that we have somehow obtained a decomposable Sullivan model $(\Lambda V, d)$ of A over our arbitrary commutative ring \mathbb{k} . Proposition 1.9 of [11] (See also [17, p 320-2]) is valid over any commutative ring \mathbb{k} . Therefore consider the multiplication of $(\Lambda V, d)$:

$$\mu : (\Lambda V', d) \otimes (\Lambda V'', d) \rightarrow (\Lambda V, d).$$

By induction on the degree of V , we can construct a factorization of μ :

$$(\Lambda V', d) \otimes (\Lambda V'', d) \xrightarrow{i} (\Lambda V' \otimes \Lambda V'' \otimes \Gamma sV, D) \xrightarrow[\phi]{\sim} (\Lambda V, d)$$

such that

- (i) $D(sv) - (v' - v'') \in \Lambda(V^{<n}) \otimes \Lambda(V^{<n}) \otimes \Gamma s(V^{<n})$ for $v \in V^n$,
- (ii) $D(\gamma^k(sv)) = D(sv)\gamma^{k-1}(sv)$ for $v \in V^{odd}$ and
- (iii) $\phi(\Gamma(sV)^+) = 0$.

Moreover, any such factorization satisfies

- (iv) i is an inclusion of CDGA's such that $(\Lambda V' \otimes \Lambda V'' \otimes \Gamma sV, D)$ is $(\Lambda V', d) \otimes (\Lambda V'', d)$ -semifree,
- (v) ϕ is a CDGA quasi-isomorphism and
- (vi) $\text{Im} D \subset (\Lambda V' \otimes \Lambda V'')^+ \otimes \Gamma sV$.

By push out in the category of CDGA's, the multiplication of A extends to a CDGA quasi-isomorphism from

$$(A \otimes A) \otimes_{(\Lambda V, d) \otimes (\Lambda V, d)} (\Lambda V' \otimes \Lambda V'' \otimes \Gamma sV, D),$$

which is $A \otimes A$ -semifree, to A . The multiplication of A also extends to a CDGA quasi-isomorphism $B(A; A; A) \xrightarrow{\sim} A$. Since \overline{A} is \mathbb{k} -semifree, the bar resolution $B(A; A; A)$ is also $A \otimes A$ -semifree. Therefore the cochains algebras

$$\begin{aligned} C(A) &= A \otimes_{A \otimes A} B(A; A; A), \\ (A \otimes \Gamma sV, \overline{D}) &= A \otimes_{A \otimes A} (A \otimes A) \otimes_{(\Lambda V, d) \otimes (\Lambda V, d)} (\Lambda V' \otimes \Lambda V'' \otimes \Gamma sV, D) \end{aligned}$$

and

$$(\Lambda V \otimes \Gamma sV, \overline{D}) = (\Lambda V, d) \otimes_{(\Lambda V, d) \otimes (\Lambda V, d)} (\Lambda V' \otimes \Lambda V'' \otimes \Gamma sV, D)$$

are weakly equivalent as CDGA's ([24, VIII.2.3], for details [28, Section 8]). So finally, we have the isomorphisms of graded algebras

$$HH_*(A) \cong H^*(A \otimes \Gamma sV, \overline{D}) \cong H^*(\Lambda V \otimes \Gamma sV, \overline{D}).$$

Proposition 8.1 *The free loop space cohomology on the complex projective space \mathbb{CP}^n , $H^*((\mathbb{CP}^n)^{S^1})$, is isomorphic as graded algebra to the Hochschild homology of $H^*(\mathbb{CP}^n)$.*

The same result (same proof) holds for the quaternionic projective space \mathbb{HP}^n .

Lemma 8.2 [2, 8.3 and 8.1g),h)]/[1, 2.1] [28, part 1. of Theorem 6.2]

- i) Let X be a simply connected CW-complex. The Adams-Hilton model of X , denoted $\mathcal{A}(X)$, can be endowed with a structure of HAH model for X .
- ii) Let $X \hookrightarrow Y$ be an inclusion of simply-connected CW-complexes. Consider an Adams-Hilton model of X , equipped with a HAH model structure for X , $\mathcal{A}(X)$. Consider an Adams-Hilton model of Y , $\mathcal{A}(Y)$. Then there is an structure of HAH model for Y on $\mathcal{A}(Y)$ that extends the HAH model structure for X of $\mathcal{A}(X)$.

Proof of Proposition 8.1

By induction on n , we suppose that the Adams-Hilton model of \mathbb{CP}^{n-1} equipped with its HAH model structure is the cobar construction $\Omega H_*(\mathbb{CP}^{n-1})$ equipped with the shuffle diagonal, denoted Δ_s . Denote by Δ the diagonal on $\Omega H_*(\mathbb{CP}^n) = \text{TA}(z_1, \dots, z_{2n-1}, d_2)$ obtained by Lemma 8.2 ii). This diagonal Δ is different from the shuffle diagonal Δ_s only on the top generator z_{2n-1} . So $(\Delta_s - \Delta)z_{2n-1}$ is a cycle. Since $\Omega \mathbb{CP}^n \approx S^1 \times \Omega S^{2n-1}$, for degree reason, it is a boundary. Therefore, we can construct a derivation homotopy from Δ_s to Δ . QED

A decomposable Sullivan model of $\frac{\mathbb{k}[x_2]}{x_2^{n+1}}$ is $(\Lambda(x_2, y_{2n+1}), d)$ with $dy_{2n+1} = x_2^{n+1}$. Using the general method described above, $HH_*(H^*(\mathbb{CP}^n))$ is the cohomology algebra of

$$\left(\frac{\mathbb{k}[x_2]}{x_2^{n+1}} \otimes \Gamma s x_2, s y_{2n+1}, \overline{D} \right)$$

with $\overline{D} s y_{2n+1} = s x(n+1)x^n$. Therefore the graded algebra $H^*((\mathbb{CP}^n)^{S^1})$ is the module

$$\begin{aligned} & \mathbb{k} \oplus \bigoplus_{1 \leq p \leq n, i \in \mathbb{N}} \mathbb{k} x^p \gamma^i(sy) \oplus \bigoplus_{0 \leq p \leq n-1, i \in \mathbb{N}} \mathbb{k} x^p s x \gamma^i(sy) \\ & \oplus \bigoplus_{i \in \mathbb{N}} \frac{\mathbb{k}}{(n+1)\mathbb{k}} x^n s x \gamma^i(sy) \oplus \bigoplus_{i \in \mathbb{N}^*} ({}_{n+1}\mathbb{k}) \gamma^i(sy) \end{aligned}$$

equipped with the obvious products. When $\mathbb{k} = \mathbb{Z}$, this is exactly Proposition 15.33 of [9] (Set $\alpha_i = x\gamma^i(sy)$ and $\beta_i = sx\gamma^i(sy)$ to make the correspondence). In particular, if $n+1 = 0$ in \mathbb{k} , we obtain the isomorphism of graded algebras

$$H^*((\mathbb{CP}^n)^{S^1}) \cong H^*(\mathbb{CP}^n) \times H^*(\Omega\mathbb{CP}^n).$$

To compute the Hochschild cohomology of an universal enveloping algebra of a Lie algebra is equivalent as to compute the Hochschild homology of a commutative algebra:

Consider a differential graded Lie algebra (in the sense of [16, 1.1(i)]) L such that $L = \{L_i\}_{i \geq 1}$ is \mathbb{k} -free of finite type. The universal enveloping algebra of L , denoted UL , has a natural structure of DGH [16, 1.1(ii)]. If $\frac{1}{2} \in \mathbb{k}$, the reduced bar construction $B(UL)$ contains a quasi-isomorphic sub-DGC $\mathcal{C}_*(L) = (\Gamma sL, d_1 + d_2)$. Its dual, denoted $\mathcal{C}^*(L)$, is a CDGA called the (*reduced*) *Cartan-Chevalley-Eilenberg complex*. Since $\mathcal{C}_*(L)$ is cocommutative, the cobar construction $\Omega\mathcal{C}_*(L)$ equipped with the shuffle diagonal is a DGH. The composite of natural DGA quasi-isomorphisms

$$\Omega\mathcal{C}_*(L) \xrightarrow{\cong} \Omega B(UL) \xrightarrow{\cong} UL$$

is a DGH morphism. By Theorem 6.2, we get immediatly

Lemma 8.3 *Suppose that $\frac{1}{2} \in \mathbb{k}$. Let L be a differential graded Lie algebra such that $L = \{L_i\}_{i \geq 1}$ is \mathbb{k} -free of finite type. Then there is a natural isomorphism of commutative graded algebras*

$$HH^*(UL) \cong HH_*(\mathcal{C}^*(L)).$$

We give now a large class of spaces who admit the universal enveloping algebra of a differential graded Lie algebra as an HAH model:

Let $r \geq 1$ be a fixed integer. Let $p \geq 2$ be an integer (eventually infinite) such that $\frac{1}{(p-1)!} \in \mathbb{k}$. Consider a r -connected CW-complex X of finite type and of dimension $\leq rp$. We want to compute the free loop space cohomology algebra $H^*(X^{S^1})$. If $p = 2$ then, by Freudenthal Suspension Theorem and Proposition 27.5 of [14], X is the suspension of a co-H space. And so we have already seen in Section 7, particularly using Corollary 7.12, how to compute its free loop space cohomology algebra $H^*(X^{S^1})$. Therefore, we can suppose that $p \neq 2$. The Adams-Hilton model of X , $\mathcal{A}(X)$, is a free chain algebra (TAV, d) on a \mathbb{k} -free graded module V concentrated in degrees between r and

$rp - 1$, endowed with a structure of HAH model for X . Therefore, by a deep Theorem of Anick [2, 5.6], there exists a free graded submodule $W \subset \text{TAV}$ such that $d(W)$ embeds into the free graded Lie algebra generated by W , $\mathbb{L}W \subset \text{TAV}$ and such that the DGA morphism

$$U(\mathbb{L}W, d) \xrightarrow{\cong} (\text{TAV}, d)$$

is an HAH isomorphism. This free differential graded Lie algebra $(\mathbb{L}W, d)$ is the model $\mathbf{L}(X)$ of Construction 8.4 of [2]. By Lemma 8.3, we have

Theorem 8.4 *With the above hypothesis and notations, there is a natural isomorphism of graded algebras*

$$H^*(X^{S^1}) \cong HH_*(C^*(\mathbf{L}(X))).$$

This Theorem extends the rational case done by [32] and was the first motivation of this paper after [28].

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